The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Numbers and Groups

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (i) Solve the simultaneous congruences

$$x \equiv 17 \pmod{102},$$
$$x \equiv 51 \pmod{76};$$

showing your working and giving your final answer in the form $x \equiv a \pmod{m}$. ***(5 marks)***

(ii) What is the remainder upon dividing 3^{1008} by 101? Show your working. ***(3 marks)***

2 Let $G_0, G_1, \ldots$ be the sequence defined by $G_0 = 1$, $G_1 = 4$, and $G_{n+2} = 6G_{n+1} + 4n^2G_n$ for all $n \geq 0$.

(i) Calculate G_2 and G_3. ***(2 marks)***

(ii) Prove, for all integers $n \geq 0$, that $G_{n+1} \geq n!2^n$ by (strong) induction. ***(4 marks)***

(iii) What is the remainder upon dividing G_{100} by 3? ***(2 marks)***

3 (i) Let $\alpha = (123)(234)(456) \in S_6$.

(a) Write α as a product of disjoint cycles. ***(1 mark)***

(b) Decide if α is even or odd. ***(1 mark)***

(c) Write down all elements in the subgroup $H = \langle \alpha \rangle \subset S_6$. ***(2 marks)***

(d) How many left cosets for H are there in S_6? ***(1 mark)***

(ii) Let $G = \mathbb{Z}_2 \times D_4$.

(a) Write down all elements of G and their orders. ***(2 marks)***

(b) Check if G is isomorphic to D_8 or not. ***(1 mark)***

4 (i) Let G, H be groups and $f : G \to H$ be a homomorphism.

Given an element $h \in H$, define a subset of G by

$$H_h = \{g \in G : f(g) = h\}.$$

Prove that H_h is a subgroup of G if and only if $h = e$.

Hint: You may find it useful to know that $f(e) = e$ and $f(g^{-1}) = f(g)^{-1}$ for all $g \in G$ whenever f is a homomorphism $f : G \to H$ (you do not need to prove this).

(4 marks)

(ii) In the conference room of the Big Corporation, seven chairs are put symmetrically at a round table, and the Health and Safety department of the Big Corporation is making a list of allowed seating plans.

The Health and Safety rules and regulations define a meeting as a gathering consisting of at least one person. By the COVID-19 restrictions, no two people can sit in adjacent chairs around the table.

With MAS114 on your CV, the Big Corporation has hired you to be their expert on symmetries and group actions. The Health and Safety department has told you that there are 28 seating plans which are allowed (you can use this and you do not have to prove this).

(a) List all types of distinct seating plans up to rotations and reflections. ***(1 mark)***

(b) For every $g \in D_7$ compute the number of seating plans fixed by g. ***(2 marks)***

(c) Use (b) and Orbit Counting to verify the number you have found in (a). ***(1 mark)***

End of Question Paper