The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Level 1 Engineering Mathematics
MAS140/151/152/153/156/161

This is an open book exam.

Answer ALL questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 130 minutes (2 hours 10 minutes) of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 A moving point has coordinates at time t given parametrically by

$$x = e^t, \qquad y = \sin t.$$

(i) Find the gradient $\dfrac{dy}{dx}$ of its trajectory as a function of t. ***(2 marks)***

(ii) Show that its trajectory satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

(4 marks)

2 In this question, define $\zeta = e^{2\pi i/7}$.

(i) Using the formula for the sum of a geometric progression, show that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^5 + \zeta^6 = 0.$$

(2 marks)

(ii) Show from (i) that

$$\left(\zeta + \zeta^2 + \zeta^4\right)^2 + \left(\zeta + \zeta^2 + \zeta^4\right) + 2 = 0.$$

(2 marks)

(iii) Deduce from (ii) the value of

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right).$$

(2 marks)

3 (i) By integrating by parts, show that

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

If necessary, you may use the integral $\int \ln x \, dx = x \ln x - x + C$. Here C is an arbitrary constant. *(2 marks)*

(ii) If $u = \dfrac{e^x}{x}$, give a simplified expression for $\ln u$.

Hence use the substitution $u = \dfrac{e^x}{x}$ to find

$$\int \frac{e^{2x}(x-1)(x - \ln x)\, dx}{x^3}.$$

(4 marks)

4 (i) Write down a diagonal 2×2 matrix with eigenvalues 3 and 7, and give the corresponding eigenvectors. ***(2 marks)***

(ii) Give an example of a 2×2 matrix with eigenvalues 3 and 7, and with eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ respectively. ***(2 marks)***

(iii) Give an example of a 2×2 matrix with only one eigenvalue 3, and only one eigenvector (up to scalar multiples). ***(2 marks)***

5 Let y be a function of x satisfying the differential equation

$$\frac{d^2y}{dx^2} - b^2 y = 2be^{bx}$$

such that, when $x = 0$, we have $y = \frac{dy}{dx} = 1$. Here $b > 0$ is a positive constant.

(i) Let $Y(s) = \mathcal{L}(y(x))$ be the Laplace transform of y. Write down an equation satisfied by $Y(s)$. ***(2 marks)***

(ii) By solving this equation and performing an inverse Laplace transform, solve the differential equation. ***(4 marks)***

End of Question Paper

MAS140/151/152/153/156/161 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \text{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x + y) + \cos(x - y)$

$2\sin x \sin y = \cos(x - y) - \cos(x + y)$

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2\tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\text{sech} x = \frac{1}{\cosh x}$$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\text{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1$$

Series

Sum of an arithmetic series:

$$\frac{\text{first term } + \text{ last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \quad (-1 < x \le 1)$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a > 0,\ a \neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \mathrm{sech}^2 x\,dx = \tanh x$$

$$\int \mathrm{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \frac{1}{a}\tanh^{-1}\frac{x}{a} \quad \text{if } |x| < a\right)$$

$$\int \mathrm{cosec}\,x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln(\mathrm{cosec}\,x - \cot x)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \quad \text{or} \quad \ln(\sec x + \tan x)$$

$$\int \mathrm{cosech}\,x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

Integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\dfrac{1}{s-a}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$