The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Algebra

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 2 hours and 15 minutes of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 Let S_6 denote the group of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Consider the elements $\alpha = (13)(246)$ and $\beta = (154)(36)$ of S_6.

(i) Compute α^2 and α^3. Determine the order of α. ***(4 marks)***

(ii) Express $\beta\alpha\beta^{-1}$ as a product of disjoint cycles. ***(3 marks)***

(iii) What is the size of the conjugacy class of α in S_6? ***(5 marks)***

(iv) What is the size of the centralizer of α in S_6? ***(3 marks)***

(v) Write down a non-identity element of the centralizer of α in S_6 that is different from α. ***(2 marks)***

2 (i) Explain why $\mathbb{F}_{17} := \mathbb{Z}/17\mathbb{Z}$ is a field, stating clearly any theorem that you use. What is the multiplicative inverse of $\overline{7}$ in $\mathbb{F}_{17}$? ***(4 marks)***

(ii) In the quotient ring $R := \dfrac{\mathbb{F}_{11}[x]}{\langle x^2 + \bar{2} \rangle}$, calculate $\overline{(x - \bar{3})(x - \bar{8})}$, expressing your answer in a form that is as simple as possible. Is R a field? Is the quotient ring $\dfrac{\mathbb{F}_7[x]}{\langle x^2 + \bar{2} \rangle}$ a field? Justify your answers. ***(10 marks)***

3 Let V be an n-dimensional vector space over $\mathbb{R}$ and let $T : V \to \mathbb{R}$ be a non-zero linear map.

(i) Explain why $\dim(\ker T) = n - 1$. ***(4 marks)***

(ii) Let U, W be subspaces of V. If $V = U + W$ and $U \cap W = \{0\}$, we say that "*V is the direct sum of U and W*" and write $V = U \oplus W$.
Show that if $\mathbf{x} \in V$ is not in the kernel of T, then $V = \ker T \oplus \operatorname{span}\{\mathbf{x}\}$.
(12 marks)

4 Let $V = C^\infty(\mathbb{R}, \mathbb{R})$ and let $U = \{f : \mathbb{R} \to \mathbb{R} \ : \ f \text{ is zero outside a bounded subset of } \mathbb{R}\}$. We define the subspace $W = V \cap U$ and equip W with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

(i) Let $D : W \to W$ be the differentiation operator on W (i.e. $Df = f'$). Assuming that it exists, show that the adjoint is $-D$. ***(6 marks)***

(ii) Is D self-adjoint? ***(2 marks)***

(iii) What about D^2? ***(2 marks)***

(iv) We say that a linear operator $T : V \to V$ is *normal* if $T^*T = TT^*$.
Is D normal? ***(3 marks)***

End of Question Paper