The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Differential Equations

This is an open book exam.

Answer ALL questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3 hours hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 For the following system of non-linear ODEs

$$\dot{u} = u - 2u^2 - uv,$$
$$\dot{v} = v - 2v^2 - uv,$$

(i) find and classify the equilibrium points, ***(7 marks)***

(ii) sketch the phase portrait for $u, v \geq 0$. ***(5 marks)***

2 Show that $x = 0$ is a regular singular point of the following equation

$$x^2y'' - 4xy' + 6y = 0. \tag{1}$$

(2 marks)

Show that for any Frobenius solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \qquad a_0 \neq 0, \tag{2}$$

α must be either 2 or 3. ***(3 marks)***

By using Equation (2) with $\alpha = 2$, find the coefficients of the Frobenius solution to Equation (1) up to and including terms of order x^5. ***(4 marks)***

Prove that there are no (non-zero) terms of order x^m for $m > 3$ when searching for solutions to Equation (1) by using Equation (2) with $\alpha = 2$. ***(3 marks)***

Write down the general solution to Equation (1). ***(1 mark)***

3 (i) Find the general form of the separable solutions of the partial differential equation (PDE)

$$2xt\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}. \tag{3}$$

(5 marks)

At time $t = 0$, the function $u(x,t)$ satisfies the initial condition $u(x,0) = \psi(x)$. Give an example of a function $\psi(x)$ for which there is no separable solution of the PDE (3) satisfying this initial condition. ***(1 mark)***

(ii) You are *given* that the separable solutions of the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{4}$$

satisfying the boundary condition

$$T(0,t) = 0, \tag{5}$$

are

$$T(x,t) = \begin{cases} A\sinh(\mu x)\, e^{\mu^2 t}, \\ Bx, \\ C\sin(\nu x)\, e^{-\nu^2 t}, \end{cases}$$

where $\nu, \mu > 0$ and A, B, C are constants.

(You do not need to derive these separable solutions.)

Find the solution of the heat equation (4) subject to the boundary condition (5) and

$$T\left(\frac{\pi}{2}, t\right) = 1 + e^{-t}.$$

(5 marks)

Are there any separable solutions of the heat equation (4) subject to the boundary condition (5) such that

$$T(x,t) \to 0 \qquad \text{as } x \to \infty?$$

Justify your answer. ***(2 marks)***

4 Find the general solution of the hyperbolic PDE

$$\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial t^2} = 3x - t.$$

(You are not required to show that the PDE is hyperbolic.) ***(12 marks)***

End of Question Paper