The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

MAS275 Probability Modelling **2 hours 30 minutes**

This is an open book exam.

Answer ***ALL*** *questions.*

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Let $(X_n, n \in \mathbb{N}_0)$ be a Markov chain on $\{0, 1\}$ with the following transition matrix:

$$\begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix},$$

and which always starts at 0: $X_0 = 0$.

(i) Find $P(X_1 = 0)$ and $P(X_2 = 0)$. ***(3 marks)***

(ii) Find the unique invariant distribution for this Markov chain. ***(2 marks)***

(iii) Justify why $\lim_{n\to\infty} P(X_n = 0) = \frac{3}{7}$. ***(3 marks)***

(iv) For $n \in \mathbb{N}_0$, what is $P(X_{n+2} = 1 \mid X_{n+1} = 1, X_n = 0)$? ***(2 marks)***

(v) For $n \in \mathbb{N}_0$, what is $P(2X_{n+2} + X_{n+1} = 3 \mid 2X_{n+1} + X_n = 2)$? ***(2 marks)***

(vi) The above reasoning, generalised, can show that the process Y_n defined by $Y_n = 2X_{n+1} + X_n$ is a Markov chain. Write down its transition matrix. (Note that $Y_n \in \{0, 1, 2, 3\}$.) ***(6 marks)***

(vii) Find $\lim_{n\to\infty} P(Y_n = 0)$. ***(5 marks)***

2 Our hero Calvin has landed on the dangerous Zorg planet. He has identified three places where he can sleep at night, labelled A, B, C. After each night at one spot, Calvin chooses one of the other two for his next night, at random. Unfortunately, unbeknownst to him, spots B and C are relatively close to a heavily patrolled Zorg area. Each time he sleeps at location B, Calvin has a 1/2 chance of being captured at the end of the night, and the probability is 1/3 if he sleeps at location C. Luckily, location A is in fact safe and Calvin will not get captured when he sleeps there.

Assuming he spends his first night at location A, what is the expectation of the number of full nights Calvin will spend before being captured? ***(9 marks)***

3 We model the arrival of earthquakes on an island over a period of two years on the interval $[0, 2]$ by a Poisson process with variable rate $\lambda(t) = t$. We let X be the number of earthquakes which arrive in the first year ($t \in [0, 1]$) and Y the number which arrive in the second year ($t \in [1, 2]$).

(i) What are the distributions of X and Y ? ***(3 marks)***

(ii) Find $E(X^2Y)$. ***(3 marks)***

4 Let $k > 1$. We are interested in a non-delayed renewal process for which the probability u_n of a renewal at time $n \in \mathbb{N}_0$ is given by

$$u_n = \frac{1}{k+1} + \frac{k}{k+1}\left(-\frac{1}{k}\right)^n .$$

Note that $u_0 = 1$: as usual there is by convention a renewal at time 0.

(i) Prove that the generating function $U(s)$ defined for $s \in [0,1)$ by $U(s) = \sum_{n=0}^{\infty} u_n s^n$, is given by

$$U(s) = \frac{k-(k-1)s}{(1-s)(k+s)}$$

(5 marks)

(ii) What is $\lim_{s \to 1^-} U(s)$? Deduce from this whether the process is recurrent or transient. ***(2 marks)***

(iii) We let f_n be the probability that the first renewal, discounting the one at time 0, occurs at time n. Deduce from (i) the expression of f_n for $n \in \mathbb{N}$.
(5 marks)

End of Question Paper