The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020-2021**

Mechanics and Fluids **2 hours 30 mins**

This is an open book exam.

Answer ALL questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours and 30 minutes of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Sketch level curves for $\phi = \mathrm{e}^x y$ and add a couple of vectors indicating $\nabla\phi$ to the sketch.

Calculate the rate of change of ϕ in the direction $\mathbf{u} = (3, 4)$ at the point $(1, 2)$.

(6 marks)

2 $\mathbf{u} = (0,\, x^2 y,\, xz^2)$.

Calculate $\nabla \times \mathbf{u}$, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and $\nabla^2 \mathbf{u}$.

(6 marks)

3 *Using suffix notation*, show that $\nabla \times (\psi \nabla \phi) = \nabla \psi \times \nabla \phi$, where ψ and ϕ are scalar functions of position $\mathbf{r}$.

(6 marks)

4 A force is given by $\mathbf{F} = (x^2 - y^2)\mathbf{i} - 2xy\mathbf{j}$. By evaluating the line integral, calculate the work done W by the force on a particle along the path consisting of two straight lines, ACB, where A$= (1, 2)$, C$= (1, 1)$ and B$= (3, 1)$.

Show that a potential exists for this force, and use it to verify W.

(8 marks)

5 A uniform lamina of mass m lies in the region $0 \leq x \leq \pi/2$, $-\cos x \leq y \leq \cos x$. Calculate the moment of inertia of the laminar about the y-axis.

(6 marks)

6 A volume is enclosed by the surfaces $x^2 + y^2 = a^2$, $z = 0$ and $z = h$. For the case $\mathbf{F} = 2yz\mathbf{j}$, evaluate $\mathbf{F} \cdot \delta\mathbf{S}$ for each of the surfaces of the volume. Hence for the full closed surface, evaluate the integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}.$$

(6 marks)

7 In cylindrical polar coordinates, the flow around a cylinder is given by

$$\mathbf{u} = \hat{\mathbf{r}}\, U(r - a^2/r)\cos(2\theta) - \hat{\boldsymbol{\theta}}\, Ur\sin(2\theta).$$

Calculate a streamfunction for the flow. Hence sketch the flow.

(6 marks)

8 The velocity potential for a point source at the origin is given in spherical polar coordinates by

$$W = -\frac{m}{4\pi}\frac{1}{r}.$$

Flow into the corner region of a large container is modelled by a point source at $\mathbf{i} + \mathbf{j}$ in the region $x \geq 0$, $y \geq 0$, $z \leq 0$, where the planes at $x = 0$ and $y = 0$ are solid walls, and $z = 0$ is the surface of the fluid. By placing imaginary sources outside the region to ensure that the appropriate conditions hold on the wall boundaries, determine the x-component of the velocity along the positive x-axis ($x \geq 0$, $y = z = 0$).

(6 marks)

End of Question Paper

VECTOR CALCULUS IDENTITIES

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{E.1}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{E.2}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{E.3}$$

$$\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi \tag{E.4}$$

$$\nabla \cdot (\phi\,\mathbf{v}) = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\phi \tag{E.5}$$

$$\nabla \times (\phi\,\mathbf{v}) = \phi\,\nabla \times \mathbf{v} + \nabla\phi \times \mathbf{v} \tag{E.6}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \tag{E.7}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) \tag{E.8}$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) \tag{E.9}$$

- (E.1)-(E.3) express the linearity property of the vector operators.
- (E.4)-(E.7) follow immediately using subscript notation and the product rule. You should know or be able to derive them quickly, e.g.

  $$\nabla \cdot (\phi\mathbf{v}) = \partial_i(\phi v_i) = \phi\,\partial_i v_i + v_i \partial_i \phi = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\phi\,.$$

- (E.8)-(E.9) you should be able to derive, given the identity

  $$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\ \ .$$

OPERATORS IN CYLINDRICAL POLARS (CPs)
AND SPHERICAL POLARS (SPs)

1. CPs (Cylindrical Polars) $\quad (r,\theta,z) \quad h_1=1,\ h_2=r,\ h_3=1$

$$\nabla V = \frac{\partial V}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\,\hat{\boldsymbol{z}} \tag{CP.1}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\,\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial\theta} + \frac{\partial F_3}{\partial z} \tag{CP.2}$$

$$\nabla\times\mathbf{F} = \left[\frac{1}{r}\frac{\partial F_3}{\partial\theta} - \frac{\partial F_2}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial r}\right]\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rF_2) - \frac{1}{r}\frac{\partial F_1}{\partial\theta}\right]\hat{\boldsymbol{z}} \tag{CP.3}$$

$$\nabla^2 V = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2} \tag{CP.4}$$

2. SPs (Spherical Polars) $\ (r,\theta,\phi) \quad h_1=1,\ h_2=r,\ h_3=r\sin\theta$

$$\nabla V = \frac{\partial V}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\,\frac{\partial V}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial\phi}\,\hat{\boldsymbol{\phi}} \tag{SP.1}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\,\{(\sin\theta)F_2\} + \frac{1}{r\sin\theta}\,\frac{\partial F_3}{\partial\phi} \tag{SP.2}$$

$$\begin{aligned}\nabla\times\mathbf{F} = \frac{1}{r\sin\theta}\,\left[\frac{\partial}{\partial\theta}\{(\sin\theta)F_3\} - \frac{\partial F_2}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r\sin\theta}\left[\frac{\partial F_1}{\partial\phi} - \frac{\partial}{\partial r}\{(r\sin\theta)F_3\}\right]\hat{\boldsymbol{\theta}}\\ +\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2) - \frac{\partial F_1}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\end{aligned} \tag{SP.3}$$

$$\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \tag{SP.4}$$