The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Fluid Mechanics I

This is an open book exam.

*Answer **both** questions. The marks awarded to each section of question are shown in italics. The total mark for the paper is 50.*

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours 30 min of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (a) Find the material derivative $D\mathbf{u}/Dt$, where $\mathbf{u}$ is the fluid velocity, for the following flows. In each case (x, y, z) are Cartesian coordinates.

(i) $\mathbf{u} = \Omega(-y, x, 0)$ where Ω is a constant. *(3 marks)*

(ii) $\mathbf{u} = \left(-\dfrac{ky}{x^2+y^2}, \dfrac{kx}{x^2+y^2}, 0\right)$ for $x^2 + y^2 \neq 0$,

where k is a constant. *(5 marks)*

(b) An incompressible viscous fluid flows outside a stationary solid cylinder whose cross-section is a circle of radius a. In cylindrical polar coordinates (r, θ, z), where the z-axis is the axis of the cylinder, the velocity $\mathbf{u}$ in the fluid is given by

$$\mathbf{u} = u_r(r)\hat{\mathbf{r}} + u_\theta(r)\hat{\boldsymbol{\theta}} + u_z(r)\hat{\mathbf{z}}$$

for some functions u_r, u_θ and u_z.

State what $u_r(a)$, $u_\theta(a)$ and $u_z(a)$ must equal, giving a reason. *(2 marks)*

Derive the result $u_r(r) = 0$. *(3 marks)*

(c) An incompressible viscous fluid flows between parallel plane solid boundaries at $y = 0$ and $y = h$, with velocity $\mathbf{u}(x, y, z)$, where (x, y, z) are Cartesian coordinates. The boundary at $y = 0$ is stationary, and the boundary at $y = h$ moves with velocity $(U, 0, 0)$, where U is a non-zero constant.

For each of the following velocity fields, determine whether the velocity field is incompressible and also whether it satisfies the no-slip conditions at the solid boundaries. *(5 marks)*

(i) $\mathbf{u} = \left(\dfrac{Uy}{h}, 0, 0\right)$

(ii) $\mathbf{u} = \left(\dfrac{Uy}{h}, 0, \dfrac{Uy}{h}\right)$

(iii) $\mathbf{u} = \left(\dfrac{Uy^2}{h^2}, \dfrac{Uy(h-y)}{h^2}, 0\right)$

(iv) $\mathbf{u} = \left(U\sin\left(\dfrac{\pi y}{2h}\right), 0, U\sin\left(\dfrac{\pi y}{h}\right)\right)$

(d) In spherical polar coordinates (r, θ, ϕ) a flow of incompressible fluid has velocity $\mathbf{u}$ given by

$$\mathbf{u} = \frac{U}{a^2}(a^2 - r^2)\cos\theta\,\hat{\mathbf{r}} + \frac{U}{a^2}(2r^2 - a^2)\sin\theta\,\hat{\boldsymbol{\theta}} \qquad \text{for } r \leqslant a,$$

where a and U are non-zero constants.

Calculate the vorticity for $r \leqslant a$, and state whether the flow is irrotational there. *(7 marks)*

2 A solid sphere of radius a and centre O is moving with constant velocity $\mathbf{U}$ in an incompressible viscous fluid. You are given that the velocity of this fluid around the sphere takes the form

$$\mathbf{u} = f(r)\mathbf{U} + g(r)(\mathbf{U}\cdot\mathbf{x})\mathbf{x}$$

for some functions f and g, where $\mathbf{x} = (x_1, x_2, x_3)$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Here $x_1 = x$, $x_2 = y$, $x_3 = z$ and (x, y, z) are Cartesian coordinates.

You are also given that

$$f(a) = 1, \qquad g(a) = 0.$$

(a) Show that

$$\nabla\cdot\mathbf{u} = h(r)\mathbf{U}\cdot\mathbf{x},$$

for some function h, where you should give h in terms of f and g.

$\left[\text{You may assume that } \dfrac{\partial r}{\partial x_j} = \dfrac{x_j}{r}.\right]$ ***(7 marks)***

(b) If

$$f(r) = A\frac{a}{r} + B\frac{a^3}{r^3}, \qquad g(r) = C\frac{a}{r^3} + D\frac{a^3}{r^5}$$

for some constants A, B, C and D, derive the values of A, B, C and D. ***(10 marks)***

(c) Use the values of A, B, C and D from part (b) to show that the vorticity $\boldsymbol{\omega}$ in the fluid satisfies

$$\boldsymbol{\omega} = \frac{3a}{2r^3}\mathbf{U}\times\mathbf{x}.$$ ***(8 marks)***

End of Question Paper