The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Quantum Theory

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2h 30mins of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 (i) Calculate the energy of the quantum state represented by the wavefunction

$$\psi(x,t) = \exp\left(\frac{m\omega x^2}{2\hbar}\right)\exp\left(\frac{-i\omega t}{2}\right),$$

and find the potential $V(x)$ for which this wavefunction is a solution of the time-dependent Schrödinger equation. ***(7 marks)***

Is ψ a realistic wavefunction? Give a reason for your answer. ***(3 marks)***

(ii) Let the Hamiltonian of a quantum system have the form

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$$

where a, b, c are real numbers. You are given the following eigenvectors of the Hamiltonian: $X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(a) The eigenvalue for X_1 is $E_1 = a + b$, for X_2 the eigenvalue is $E_2 = c$. Find the eigenvalue for the vector X_3. ***(3 marks)***

(b) Does the state

$$X(t) = \alpha X_1 \exp(-i(a+b)t/\hbar) + \beta X_2 \exp(-i(a+b+c)t/\hbar)$$

describe a possible evolution given the Hamiltonian H? Give a reason for your answer. ***(3 marks)***

(c) If the system is in the state $\Psi_1(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ at $t = 0$, what is the state at a later time t, i.e. what is $\Psi_1(t)$? ***(3 marks)***

(d) If the system is in the state $\Psi_2(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ at $t = 0$, what is the state at a later time t, i.e. what is $\Psi_2(t)$? ***(6 marks)***

2 A particle of mass m is moving under the influence of a potential $V(x)$ (where $V(x)$ is a real function) in one dimension and is described by a wave function $\psi(x,t)$. The wavefunction solves the Schrödinger equation and is normalised, i.e.

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 dx = 1 \ .$$

Let the expectation value of the position be denoted by $E_\psi(x)$ and let $j(x,t)$ be the probability current. In the following, do **not** use any commutation relations.

(i) Starting from definition of the expectation value

$$E_\psi(x) = \int_{-\infty}^{+\infty} \bar{\psi}(x,t) x \psi(x,t) dx \ ,$$

show that

$$\frac{d}{dt} E_\psi(x) = \int_{-\infty}^{+\infty} j(x,t) dx \ .$$

(16 marks)

(ii) By explicit calculation and without using the result (i), show that

$$\int_{-\infty}^{+\infty} j(x,t) dx = \frac{1}{m} \int_{-\infty}^{+\infty} \bar{\psi}(x,t) \hat{p} \psi(x,t) dx \ ,$$

(8 marks)

where $\hat{p}$ is the momentum operator.

Interpret the results from (i) and (ii). ***(1 mark)***

Hint: Note that $\psi(x,t)$ is square-integrable, which means that $\psi \to 0$ for $x \to \pm\infty$. Likewise, $d\psi/dx$ and $d\bar{\psi}/dx$ vanish for $x \to \pm\infty$. Assume that

$$x \psi \frac{d\bar{\psi}}{dx}$$

and

$$x \bar{\psi} \frac{d\psi}{dx}$$

vanish for $x \to \pm\infty$. Consider integration by parts in (i) and (ii).

End of Question Paper