Data provided: Formulae sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020-2021**

Mathematics (Computational Methods) **2 hours and 30 minutes**

This is an open book exam.

Answer ALL questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours and 30 minutes of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic:

(a) $$7\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} = \sin u$$ ***(1 mark)***

(b) $$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} = 3$$ ***(1 mark)***

(c) $$2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$ ***(1 mark)***

(ii) For the second order partial differential equation

$$12\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial^2 \phi}{\partial x \partial y} + 3\frac{\partial^2 \phi}{\partial y^2} + 5\frac{\partial \phi}{\partial x} = 6,$$

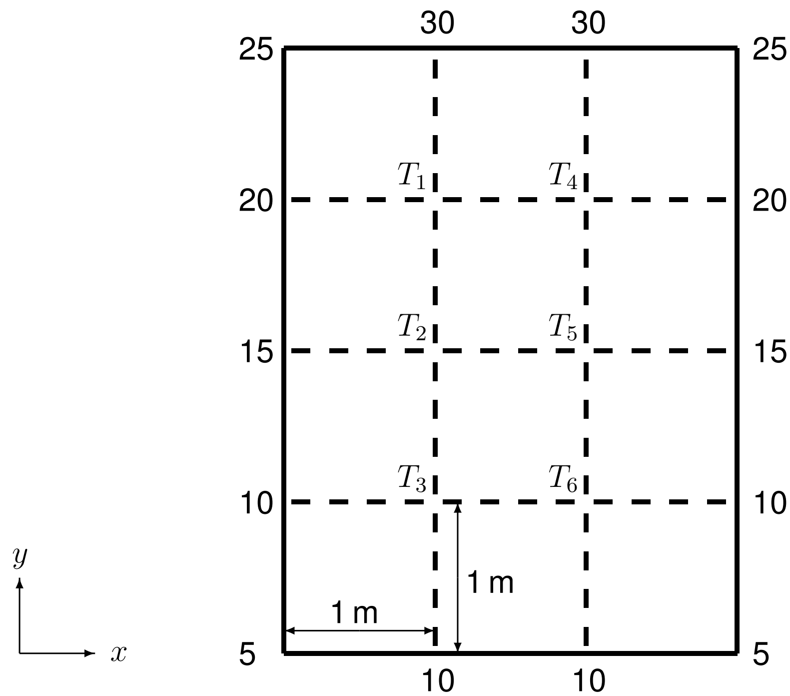
where $\phi = \phi(x, y)$ and b is a constant, find the range of values of b such that the differential equation is (a) parabolic, (b) hyperbolic and (c) elliptic.
(3 marks)

 Question 1 continued on next page

1 (continued)

(iii) The figure shows a rectangular plate made of a homogeneous isotropic material. The plate is divided into intervals of equal length 1 m in the x and y directions. The temperature $T(x, y)$ in this plate satisfies the indicated boundary conditions (given in °C) and has reached a steady-state condition so that it is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$



(a) Draw a sketch of the solution domain, showing clearly the line of symmetry for $T(x, y)$, and indicating which of the unknown temperatures $T_1, \ldots, T_6$ are equal to each other. ***(2 marks)***

(b) Derive the finite difference scheme required to find estimates of the nodal temperatures T_1, T_2 and T_3, subject to the boundary conditions indicated on the diagram. ***(8 marks)***

(c) Express these finite difference equations in the form $A\mathbf{T} = \mathbf{b}$, where A is a 3×3 matrix, $\mathbf{T} = (T_1, T_2, T_3)^T$ and $\mathbf{b} = (50, 15, 20)^T$, with units in °C, where you should give the matrix A. Solve these matrix equations equations to find the components of $\mathbf{T}$.

(9 marks)

2 (i) (a) Find and classify the stationary points of the function

$$f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 3.$$

(6 marks)

(b) Calculate an approximation to the minimum of

$$h(x,y) = (x-y)^2 + \frac{1}{16}(x+y+1)^4$$

using one iteration of Newton's method starting from the point $(0,0)$. ***(6 marks)***

(c) Let

$$p(x,y) = 4(x-0.6)^2 + 3(y-0.3)^2.$$

Starting from the point $(1.0, 1.0)$ apply the method of steepest descent for one iteration. Do your workings correct to FOUR decimal places. ***(7 marks)***

(ii) A company plans to build **six** schools on selected sites during the next **five** years. The construction of each school is completed within one year, where each period of one year starts on 1 January and ends on 31 December. Let

$$x_{ij} = \begin{cases} 1 & \text{if school } i \text{ is constructed in year } j \\ 0 & \text{if school } i \text{ is not constructed in year } j \end{cases}$$

and c_{ij} = cost of building school i in year j.

Set up the integer programming problem to minimise the total costs over the five year period provided all the schools are built. ***(2 marks)***

For each of the **separate** cases below, list the additional constraints required.

(a) Schools 2, 4, 5 and 6 must be built by the end of year 4. ***(1 mark)***

(b) In the first three years either schools 2 and 3, or, schools 5 and 6 must be built. ***(3 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\, t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$