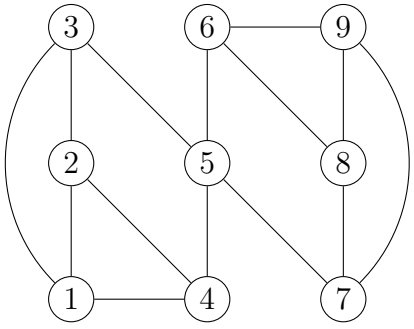
The
University
Of
Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

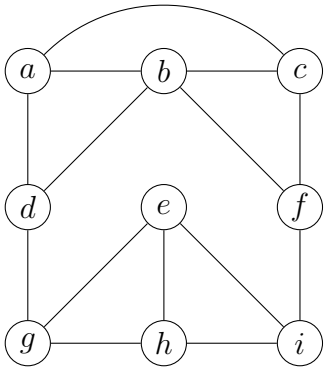
Graph Theory

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

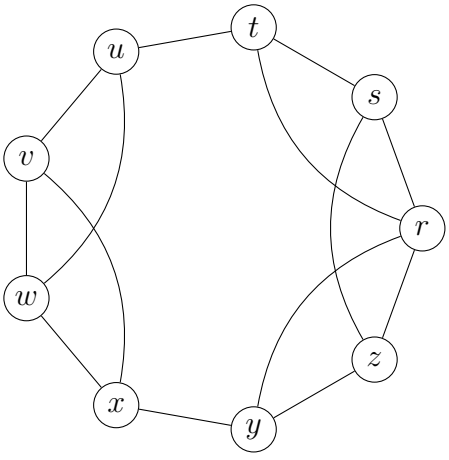
1 This question concerns the four graphs Γ_1 to Γ_4 shown below.



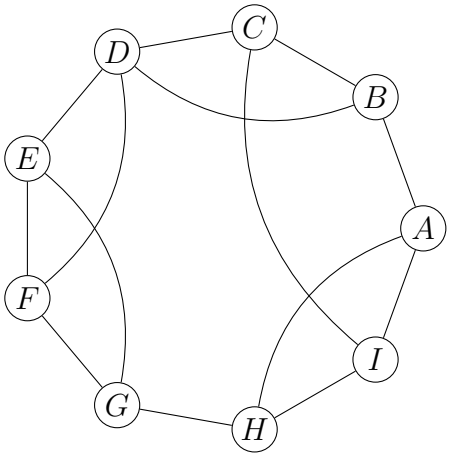
Γ_1



Γ_2



Γ_3



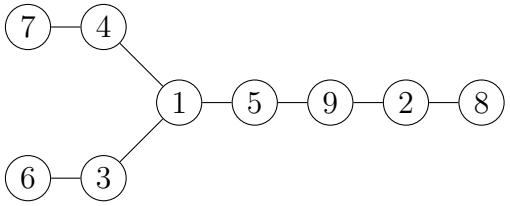
Γ_4

(i) All of the Γ_i have the same degree sequence – what is it? Are any of the Γ_i semi-Eulerian? ***(3 marks)***

(ii) Exactly one of the Γ_i is not Hamiltonian. Which one? Justify your answer, making sure to show the other graphs *are* Hamiltonian. ***(4 marks)***

(iii) Which of the Γ_i are isomorphic to each other, and which aren't? Justify your answer. ***(5 marks)***

2 (i) Give the Prüfer code of the following labelled tree:



(4 marks)

The table below shows the costs of travelling between 8 different towns, A-H.

A							
8	B						
7	10	C					
1	2	9	D				
1	6	9	10	E			
4	4	7	4	3	F		
8	7	5	7	3	2	G	
10	10	5	9	2	7	10	H

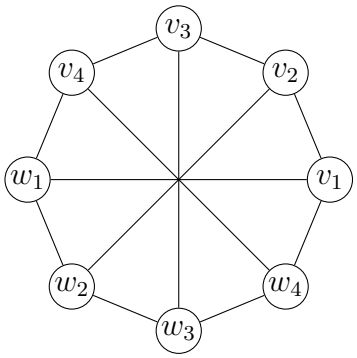
(ii) When using the nearest neighbour algorithm, sometimes two edges will have the same weight, and an arbitrary choice needs to be made. For example, if we started the nearest neighbour algorithm at C, we could first go to either G or H. Find all possible Hamiltonian cycles that can be produced running the nearest neighbour algorithm starting from A. Which of them give the best bound on the Travelling Salesperson Problem? ***(5 marks)***

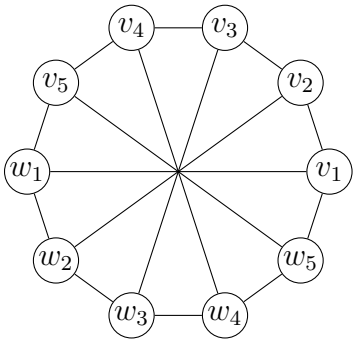
(iii) Following the method covered in lecture, obtain a lower bound on the Travelling Salesperson Problem, starting by deleting the vertex A. ***(4 marks)***

3 In this question and the next, the graph O_{2n} refers to the trivalent graph with $2n$ vertices and $3n$ edges that is obtained by taking the cycle graph C_{2n} and adding an extra edge between each vertex and its "opposite" vertex in the cycle.

More precisely, (and introducing notation you may find helpful), O_{2n} is the graph on vertices $v_1, \ldots, v_n$ and $w_1, \ldots w_n$ where for $1 \leq i \leq n-1$ we have v_i is adjacent to v_{i+1}, w_i is adjacent to w_{i+1}, and v_i is adjacent to w_i, and in addition v_n is adjacent to w_1, w_n is adjacent to v_1, and w_n is adjacent to v_n.

The graphs O_8 and O_{10} are shown below, with vertices labeled as described in the previous paragraph.





(i) Using Kuratowski's Theorem, prove that O_{2n} is not planar for $n \geq 3$. Further, explain how your work shows that it is possible to remove $n-3$ edges from O_{2n} and still have a nonplanar graph. ***(5 marks)***

(ii) Draw O_{2n} on the Mobius band without edges crossing. ***(3 marks)***

(iii) Using the Planarity Algorithm for Hamiltonian Graphs, show that it is possible to remove a single edge e from O_{2n} and produce a planar graph. You will need to make a careful choice of Hamiltonian cycle. Using the output from the planarity algorithm or otherwise, give a planar drawing of $O_{2n} \setminus e$ for some edge e. ***(5 marks)***

4 This question also concerns the graph O_{2n} with $2n$ vertices defined in Question 3.

(i) Determine, with justification, the chromatic number and chromatic index of O_4. ***(3 marks)***

(ii) For $n \geq 3$, determine the chromatic index $\chi'(O_{2n})$ of O_{2n}. Be sure to justify your answer. The answer is independent of n. ***(3 marks)***

(iii) For $n \geq 3$, determine the chromatic number $\chi(O_{2n})$ of O_{2n}. Be sure to justify your answer. The answer depends on whether n is odd or even. ***(6 marks)***

End of Question Paper