Data provided: Supplementary Formula Sheet

MAS352

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Stochastic Processes and Financial Mathematics **3.5 hours**

*Candidates should attempt **ALL** questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 50.

This is an open book exam.

*You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). **Late submission will not be considered without extenuating circumstances.***

Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Let $\Omega = \{-2,-1,0,1,2\}$. Consider the σ-fields (a)-(c) defined as follows.

(a) $\{\emptyset, \{-2,-1,0\}, \{1,2\}, \Omega\}$

(b) $\sigma(\{-2,0,2\}, \{0\})$

(c) $\sigma(\{-2,-1\}, \{-2,-1,0\}, \{-1,1,2\})$

For each of the σ-fields (a)-(c), write down which items of information listed as (i)-(v) are contained within the information characterized by that σ-field. Some σ-fields may contain more than one item of information.

(i) Whether the outcome is non-negative.

(ii) Whether the outcome is strictly positive.

(iii) Whether the outcome is non-zero.

(iv) Whether the outcome has absolute value 1.

(v) Whether the outcome is equal to -1.

(5 marks)

2 Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Explain carefully why

$$Y = \begin{cases} X^2 & \text{if } X > 1 \\ 1 & \text{if } X \leq 1 \end{cases}$$

is also a random variable. ***(4 marks)***

You may use standard results about measurability of sums, products, limits and indicator functions, providing they are clearly stated.

3 Let $(X_i)_{i=1}^{\infty}$ be a sequence of independent, identically distributed random variables. Suppose that there exists some constant K such that

$$0 \leq X_i \leq K.$$

Let $M_n = e^{X_1+\ldots+X_n}$ and $\mu = \mathbb{E}[e^{X_1}]$.

(a) Show that (M_n) is a submartingale and that $(\frac{M_n}{\mu^n})$ is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(X_i\ ;\ i \leq n)$. ***(6 marks)***

(b) (i) Explain briefly why there exists a random variable M_∞, taking values in $[0,\infty]$, such that $M_n \stackrel{a.s.}{\to} M_\infty$.

(ii) Prove that $\mathbb{P}[M_\infty \in \{1,\infty\}] = 1$.

(4 marks)

4 This question concerns the binomial model, in discrete time, with two assets, cash and stock.

A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.

Suppose that we have $T = 2$ steps of time, and let the parameters of the model be $p_u = 0.05$, $p_d = 0.95$, $u = 1.8$, $d = 0.8$, $r = 0.2$ and $s = 100$.

Consider the contingent claim

$$\Phi(S_T) = \max(150 - S_T, 0)$$

Draw a recombining tree of the stock price process at time $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for this contingent claim, at each node, along with a portfolio strategy that hedges $\Phi(S_T)$.

(10 marks)

5 Let (B_t) be a standard Brownian motion. Let

$$M_t = e^{-t/2} \cosh(B_t),$$
$$N_t = e^{-t/2} \sinh(B_t).$$

(a) Show that (M_t) and (N_t) are both martingales. ***(4 marks)***

(b) Deduce that

$$M_t - 1 = \int_0^t \left(\int_0^s M_u \, dB_u \right) dB_s$$

and

$$N_t + B_t = \int_0^t \left(\int_0^s N_u \, dB_u \right) dB_s.$$

(4 marks)

(c) Show that

$$\mathrm{var}(M_t) = \int_0^t \int_0^s 1 + \mathrm{var}(M_u) \, du \, ds.$$

Hence find a second order ordinary differential equation satisfied by $v(t) = \mathrm{var}(M_t)$. ***(6 marks)***

6 This question concerns the Black-Scholes model, in continuous time.

A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.

Assume that $S_0 = 1$ and let $\Phi(S_T) = 1/S_T$.

(a) Find the value Π_t of the contingent claim $\Phi(S_T)$ at time $t \in [0, T]$.

(5 marks)

(b) Suppose that, at time $t = 0$, we hold a portfolio consisting of a single contract with contingent claim $\Phi(S_T)$.

How many units of stock should we buy or sell, to delta hedge our portfolio at time $t = 0$?

Is it possible for the resulting delta neutral portfolio to be gamma neutral at $t = 0$? ***(2 marks)***

End of Question Paper

MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- $X_n \stackrel{d}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[X_n \le x] = \mathbb{P}[X \le x]$ whenever $\mathbb{P}[X \le x]$ is continuous at $x \in \mathbb{R}$.
- $X_n \stackrel{\mathbb{P}}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[|X_n - X| > a] = 0$ for every $a > 0$.
- $X_n \stackrel{a.s.}{\to} X \;\Leftrightarrow\; \mathbb{P}\left[X_n \to X \text{ as } n \to \infty\right] = 1$.
- $X_n \stackrel{L^p}{\to} X \;\Leftrightarrow\; \mathbb{E}\left[|X_n - X|^p\right] \to 0$ as $n \to \infty$.

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants r (discrete interest rate), p_u and p_d (probabilities of stock price increase/decrease), u and d (factors of stock price increase/decrease), and s (initial stock price).

The value of x in cash, held at time t, will become $x(1+r)$ at time $t+1$.

The value of a unit of stock S_t, at time t, satisfies $S_{t+1} = Z_t S_t$, where $\mathbb{P}[Z_t = u] = p_u$ and $\mathbb{P}[Z_t = d] = p_d$, with initial value $S_0 = s$.

When $d < 1 + r < u$, the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \qquad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time $t = 0, 1, 2, \ldots, T$. The case $T = 1$ is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale M_n and a stopping time T, holds if any one of the following conditions is fulfilled:

(a) T is bounded.

(b) M_n is bounded and $\mathbb{P}[T < \infty] = 1$.

(c) $\mathbb{E}[T] < \infty$ and there exists $c \in \mathbb{R}$ such that $|M_n - M_{n-1}| \le c$ for all n.

MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t,x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t)\,dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t,T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Feynman-Kac formula

Suppose that $F(t,x)$, for $t \in [0,T]$ and $x \in \mathbb{R}$, satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \alpha(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\beta(t,x)^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) &= 0 \\ F(T,x) &= \Phi(x). \end{aligned}$$

If X_u satisfies $dX_u = \alpha(u, X_u)\,dt + \beta(u, X_u)\,dB_u$, then

$$F(t,x) = e^{-r(T-t)} \mathbb{E}_{t,x}\left[\Phi(X_T)\right].$$

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t\,dt$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$, with initial value S_0.

At time $t \in [0,T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0,$$
$$F(T,s) = \Phi(s).$$

The unique solution F satisfies

$$F(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)\,|\,\mathcal{F}_t]$$

for all $t \in [0,T]$. Here, the 'risk-neutral world' $\mathbb{Q}$ is the probability measure under which S_t satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0,1]$, we define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability η_j the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.