Data provided: Background information; table of χ^2 critical values

The University Of Sheffield.

MAS371

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Applied Probability **2 hours 30 minutes**

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours 30 minutes of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

There are 30 marks available on the paper.

1 A discrete time Markov chain with state space $S = \{A, B, C\}$ is assumed to have a transition matrix of the form

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-q & 0 & q \\ r & 1-r & 0 \end{pmatrix}.$$

In data containing 38 transitions of the chain, the numbers of transitions between each pair of states were as follows:

	To A	To B	To C
From A	0	6	7
From B	9	0	5
From C	3	8	0

(a) Find the maximum likelihood estimates of p, q and r given the data. *(3 marks)*

(b) Use Wilks' Theorem to approximately test the hypothesis that $p = q = r$. *(7 marks)*

2 A continuous time Markov chain with state space $\{1, 2, 3\}$ has its transition matrices $P(t)$ given by

$$\frac{1}{5}\begin{pmatrix} 2+e^{-2t}(3\cos t+\sin t) & 2+e^{-2t}(-2\cos t+\sin t) & 1+e^{-2t}(-\cos t-2\sin t) \\ 2+e^{-2t}(-2\cos t-4\sin t) & 2+e^{-2t}(3\cos t+\sin t) & 1+e^{-2t}(-\cos t+3\sin t) \\ 2+e^{-2t}(-2\cos t+6\sin t) & 2+e^{-2t}(-2\cos t-4\sin t) & 1+e^{-2t}(4\cos t-2\sin t) \end{pmatrix}.$$

(a) It can be shown that the generator matrix G of the chain is

$$G = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

By differentiating the entries of $P(t)$, verify this for the entries in the top row of G. *(2 marks)*

(b) Assume the chain is known to start in state 2. Describe the random behaviour of the chain up to and including its second change of state. *(4 marks)*

(c) Considering the evolution of $P(t)$ for large t, what would you expect the unique stationary distribution $\boldsymbol{\pi}$ of the chain to be? Explain your answer carefully. *(3 marks)*

(d) Verify that your $\boldsymbol{\pi}$ from (c) satisfies $\boldsymbol{\pi}G = \mathbf{0}$. *(1 mark)*

3 A shop monitors its sales over a five hour period, modelled as the interval $[0, 5]$. At time 2, the shop changes its sales strategy, and is interested in whether this had an effect. To investigate this, assume that the sales occur as a possibly inhomogeneous Poisson process with rate function

$$\lambda(t) = \begin{cases} \alpha & 0 < t \leq 2 \\ \beta & 2 < t \leq 5 \end{cases}$$

where the parameters α and β represent the rates of sales before and after the change respectively.

(a) Show that the log likelihood of (α, β) can be written

$$l = -(2\alpha + 3\beta) + n_2 \log \alpha + (n_5 - n_2) \log \beta + C,$$

where n_2 is the number of sales in $(0, 2]$, $n_5 - n_2$ is the number of sales in $(2, 5]$ and C is a constant. You may use the fact, given in the course notes, that the log likelihood for the rate function of an inhomogeneous Poisson process based on observations from time 0 to t is

$$l = -\Lambda(t) + \sum_{i=1}^{n} \log \lambda(v_i) + C,$$

where $\Lambda(t) = \int_0^t \lambda(s)\, ds$, n is the number of the occurrences, the v_i are the times of the occurrences, and C is a constant. ***(3 marks)***

(b) In fact, 56 customers arrive in $(0, 5]$, of whom 32 arrived in $(0, 2]$.

(i) Find the maximum likelihood estimates of α and β. ***(2 marks)***

(ii) Use Wilks' Theorem to approximately test the hypothesis that $\alpha = \beta = 10$. ***(5 marks)***

End of Question Paper

Background material for MAS371 exam

For the purposes of the MAS371 exam, you may assume these results in the somewhat simplified form they are given in this document.

Asymptotic normality of maximum likelihood estimators

Given a vector of unknown parameters $\boldsymbol{\theta}$, the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ has a distribution which is asymptotically (in the large sample limit) Normal with mean vector $\boldsymbol{\theta}_0$ and covariance matrix approximately given by $J(\hat{\boldsymbol{\theta}})^{-1}$. Here $\boldsymbol{\theta}_0$ is the true value and $J(\boldsymbol{\theta})$ is the observed information matrix.

The r, s entry of $J(\boldsymbol{\theta})$ is given by

$$J(\boldsymbol{\theta})_{rs} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \theta_r \partial \theta_s},$$

where $\ell(\boldsymbol{\theta})$ is the log likelihood.

Wilks' Theorem

Let $\boldsymbol{\theta}$ be a vector of unknown parameters with $\boldsymbol{\theta} \in \Theta$, where Θ is a p-dimensional set. Let $\boldsymbol{\theta}_0$ be the true value.

Simple hypothesis

Consider the null hypothesis $H_0 : \boldsymbol{\theta}_0 = \boldsymbol{\theta}^*$, where $\boldsymbol{\theta}^*$ is a specified value. Then, under H_0,

$$W = -2(\ell(\boldsymbol{\theta}^*) - \ell(\hat{\boldsymbol{\theta}})),$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator and ℓ is the log likelihood, has an asymptotically (in the large sample limit) χ^2 distribution with p degrees of freedom.

Composite hypothesis

Let Θ_0 be a q-dimensional subset of Θ, and consider the null hypothesis $H_0 : \boldsymbol{\theta}_0 \in \Theta_0$. Then, under H_0,

$$W = -2(\ell(\boldsymbol{\theta}^*) - \ell(\hat{\boldsymbol{\theta}})),$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator, $\boldsymbol{\theta}^*$ is the restricted maximum likelihood estimator under the null hypothesis, and ℓ is the log likelihood, has an asymptotically (in the large sample limit) χ^2 distribution with $p - q$ degrees of freedom.

Table of the q-quantile of the χ^2 distribution with ν degrees of freedom

		ν								
		1	2	3	4	5	6	7	8	9
q	0.10	0.02	0.21	0.58	1.06	1.61	2.2	2.83	3.49	4.17
	0.50	0.45	1.39	2.37	3.36	4.35	5.35	6.35	7.34	8.34
	0.90	2.71	4.61	6.25	7.78	9.24	10.64	12.02	13.36	14.68
	0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92
	0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67
		ν								
		10	20	30	40	50	60	70	80	90
q	0.10	4.87	12.44	20.6	29.05	37.69	46.46	55.33	64.28	73.29
	0.50	9.34	19.34	29.34	39.34	49.33	59.33	69.33	79.33	89.33
	0.90	15.99	28.41	40.26	51.81	63.17	74.4	85.53	96.58	107.57
	0.95	18.31	31.41	43.77	55.76	67.5	79.08	90.53	101.88	113.15
	0.99	23.21	37.57	50.89	63.69	76.15	88.38	100.43	112.33	124.12