The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Analytical Dynamics and Classical Field Theory

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3.5 hours (3 hr 30 mins) of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 (i) A particle P of mass m is constrained to move in the (x, y)-plane. The particle is connected to a spring of zero mass and zero natural length. The other end of the spring is fixed at the origin. The potential energy stored in the spring is $V = \frac{1}{2}k\ell^2$, where $k > 0$ is a constant and ℓ is the length of the spring. There are no additional forces acting.

Show that the Lagrangian of the system is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right).$$

(3 marks)

Find the specific form of Lagrange's equations governing the motion of the particle. ***(2 marks)***

Find *two* constants of the motion, justifying your answers and interpreting the constants physically. ***(6 marks)***

(ii) In an alternative theory of electromagnetism, the Lagrangian density for an electromagnetic field with electromagnetic potential A_μ is

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right).$$

Derive the field equations satisfied by the electromagnetic potential in this theory. ***(5 marks)***

Show that

$$\partial_\mu\left[A^\mu \partial_\nu A^\nu - A_\nu \partial^\nu A^\mu\right] = \left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right) - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right).$$

(3 marks)

In another theory of electromagnetism, the Lagrangian density for the electromagnetic potential is

$$\widetilde{\mathcal{L}}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right),$$

where $F_{\mu\nu}$ is the electromagnetic field. Find a contravariant vector X^μ such that

$$\mathcal{L} - \widetilde{\mathcal{L}} = \partial_\mu X^\mu.$$

(4 marks)

Hence, *without deriving the field equations directly from* $\widetilde{\mathcal{L}}$, find the form of the field equations arising from the Lagrangian density $\widetilde{\mathcal{L}}$. You may assume that any boundary terms vanish. ***(2 marks)***

2 (i) A certain vector field X^μ satisfies

$$\mathcal{L}_X g_{\mu\nu} = f(x) g_{\mu\nu}$$

where $\mathcal{L}_X$ is the Lie derivative along X^μ, and $f(x) \neq 0$ is an arbitrary function.

(a) Show that

$$\mathcal{L}_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu,$$

where ∇_μ is the metric-compatible covariant derivative. ***(1 mark)***

(b) Show that $Z \equiv u^\mu X_\mu$ is constant along an affinely-parameterized *null* geodesic with tangent vector u^μ. ***(4 marks)***

(c) Let $J^\mu = T^{\mu\nu} X_\nu$ where $T^{\mu\nu}$ is a stress-energy tensor.
Use the standard properties of the stress-energy tensor to show that J^μ is divergence-free ($\nabla_\mu J^\mu = 0$) if and only if the stress energy tensor $T^{\mu\nu}$ is traceless. ***(3 marks)***

(ii) An homogeneous, isotropic and *closed* universe has the line element

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where $a(t)$ is the scale factor and the coordinates are $x^\alpha = [t, \chi, \theta, \phi]$.

(a) By applying the Euler-Lagrange equations to the Lagrangian $L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, or otherwise, find the following components of the affine connection: ${\Gamma^\phi}_{t\phi}$, ${\Gamma^\phi}_{\chi\phi}$ and ${\Gamma^\phi}_{\theta\phi}$. ***(5 marks)***

(b) A closed universe starts with a big bang at $t = 0$ and reaches a maximum scale factor of $a = a_m$, before contracting again to a big crunch. The *trace* of the Einstein field equations yields the equation

$$a\ddot{a} + \dot{a}^2 + 1 = \frac{4\pi a^2}{3}\left(\rho - 3P\right).$$

Solve this equation under the assumption that the universe contains *only* electromagnetic radiation, to show that

$$a(t) = \sqrt{t(2a_m - t)}.$$

Sketch the scale factor $a(t)$ as a function of t. ***(7 marks)***

(c) A light ray starts at $\chi = 0$ at the beginning of the universe ($t = 0$). Find the value of χ which the ray has reached by the end of the universe ($t = 2a_m$). ***(5 marks)***

End of Question Paper