The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2020-2021**

Magnetohydrodynamics **2.5 Hours**

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (i) We are interested in the behaviours of a sunspot. It is assumed that, with all numerical values given in standard SI units, the suitable length scale for the problem is $\ell = 10^7$, the estimate of the magnetic field is $B = 0.3$, the density ρ is approximately 2×10^{-4}, the temperature is $T = 4000$, and the suitable velocity scale is $v = 10^3$. It is given that the magnetic diffusivity is $\eta = 10^3$, the kinematic viscosity is $\nu = 8$, the universal gas constant is $R = 8.3 \times 10^3$, the average atomic weight $\widetilde{\mu} \approx 0.6$, and the magnetic permeability is $\mu_0 = 4\pi \times 10^{-7}$. Gravity can be ignored in this analysis.

(a) Estimate the magnetic Reynolds number, and comment on how the induction equation can be simplified. ***(2 marks)***

(b) By estimating relevant parameters, determine how the Navier-Stokes equation can be simplified. Hence determine if the magnetic field can be satisfactorily described as a force-free field. ***(14 marks)***

(ii) Without calculation, explain concisely why the volume enclosed in a closed magnetic flux tube (in the shape of, e.g., a torus) does not change with time in incompressible ideal MHD flows. ***(4 marks)***

(iii) It is given that $\mathbf{v} = (x\cos(t), -y\cos(t), 0)$, $\mathbf{B} = (0, B_y(x,t), 0)$, and

$$B_y(x, 0) = xe^{-x^2}. \tag{1}$$

Assuming magnetic diffusion is negligible, find $B_y(x,t)$. ***(20 marks)***

(iv) The integral I is defined by

$$I = \int_V \boldsymbol{\omega} \cdot \mathbf{A} dV, \tag{2}$$

where $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$ is the vorticity, $\mathbf{A}$ is the vector potential of $\mathbf{B}$, and V is a fixed volume in space. Assuming that V extends to infinity in all directions and that both $\mathbf{v}$ and $\mathbf{B}$ decay to zero quickly at infinity, show that

$$I = \int_V \mathbf{v} \cdot \mathbf{B} dV. \tag{3}$$

By considering a velocity field induced by a single, thin, closed vortex tube and a magnetic field $\mathbf{B}$ containing a single, thin, closed magnetic flux tube, show that I characterises the linkedness between flux tubes and vortex tubes. ***(10 marks)***

2 We consider an ideal MHD flow with an additional forcing term given by $-2\mathbf{\Omega} \times \mathbf{v}$ where $\mathbf{\Omega}$ is a constant vector. Neglecting the gravity, the Navier-Stokes equation can be written as

$$\rho\partial_t \mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{J}\times\mathbf{B} - 2\rho\mathbf{\Omega}\times\mathbf{v}. \tag{4}$$

The flow is assumed to be incompressible. The magnetic field $\mathbf{B}$ is described by the usual induction equation:

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}). \tag{5}$$

It is given that, in equilibrium, the velocity $\mathbf{v} = 0$, the magnetic field is $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ with B_0 being a constant, and the density $\rho = \rho_0$ and the pressure $p = p_0$ are both constants.

(i) Show that the linearised induction equation for magnetic perturbations $\mathbf{B}_1$ is

$$\partial_t \mathbf{B}_1 = B_0 \partial_z \mathbf{v}_1, \tag{6}$$

where $\mathbf{v}_1$ is the velocity perturbation. ***(6 marks)***

(ii) Show that the linearised Navier-Stokes equation is

$$\rho_0\partial_t\mathbf{v}_1 = -\nabla(p_1 + \mu_0^{-1}\mathbf{B}_0\cdot\mathbf{B}_1) + \mu_0^{-1}(\mathbf{B}_0\cdot\nabla)\mathbf{B}_1 - 2\rho_0\mathbf{\Omega}\times\mathbf{v}_1, \tag{7}$$

where p_1 is the pressure perturbation. ***(14 marks)***

(iii) Show that $\mathbf{v}_1$ satisfies

$$\partial_t^2(\nabla\times\mathbf{v}_1) - 2(\mathbf{\Omega}\cdot\nabla)\partial_t\mathbf{v}_1 = v_A^2\frac{\partial^2(\nabla\times\mathbf{v}_1)}{\partial z^2}, \tag{8}$$

where $v_A = B_0/(\rho_0\mu_0)^{1/2}$ is the Alfven speed. ***(10 marks)***

(iv) By considering a plane wave solution $\mathbf{v}_1 = \hat{\mathbf{v}}_1 \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$, show that the dispersion relation is

$$\omega^4 - 2\left[v_A^2 k_z^2 + 2(\mathbf{\Omega}\cdot\mathbf{k})^2 k^{-2}\right]\omega^2 + v_A^4 k_z^4 = 0. \tag{9}$$

(14 marks)

(v) Find the group velocity when $B_0 = 0$. ***(4 marks)***

(vi) Describe concisely the roles of the magnetic pressure and the magnetic tension force in sustaining the wave when $\mathbf{\Omega} = 0$. ***(2 marks)***

End of Question Paper