The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Algebraic Topology **Three hours**

This is an open book exam.
Answer both questions.
You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within three hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances.

Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Give examples as follows, justifying your answers.

(a) Topological spaces X and Y, together with injective functions $f\colon X \to Y$ and $g\colon Y \to X$ such that f, $f \circ g$ and $g \circ f$ are all continuous, but g is not continuous. ***(4 marks)***

(b) A compact, path-connected space X together with a continuous map $f\colon X \to X$ with no fixed points. ***(4 marks)***

(c) A space X such that $H_1(X)$ is not a free abelian group. (Note here that the zero group is free abelian with no generators, so in particular $H_1(X)$ must be nonzero.) ***(4 marks)***

(d) A space X together with points $a, b, c \in X$ such that $|\Pi(X; a, b)| \neq |\Pi(X; b, c)|$. ***(4 marks)***

(e) A space X such that $\pi_1(X)$ is a free group with 3 generators, and $H_2(X) = \mathbb{Z}$. ***(4 marks)***

2 Fix $n \geq 2$. Define an equivalence relation on the disc $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by $z_0 \sim z_1$ iff $z_0 = z_1$, or ($|z_0| = |z_1| = 1$ and $z_0^n = z_1^n$). Put $X = B^2/\sim$ and

$$Y = \{(u, v) \in \mathbb{C}^2 \mid |u| \leq 1, \quad v^n = (1 - |u|)^n u\}.$$

Note that when $n = 2$ we just have $X = \mathbb{R}P^2$; this should guide your thinking about the general case.

(a) Show carefully that there is a homeomorphism $f : X \to Y$ such that $f([z]) = (z^n, (1 - |z|^n)z)$ for all $z \in B^2$. You should prove in particular that f is well-defined, injective and surjective, and that both f and f^{-1} are continuous. You may assume that polynomials and the absolute value function are continuous, but beyond that you should not assume any properties of the given formula without proof. ***(13 marks)***

(b) For the boundary $S^1 \subset B^2$, explain briefly why $S^1/\sim$ is homeomorphic to S^1 again. ***(3 marks)***

(c) By adapting the method used for $\mathbb{R}P^2$, calculate $H_*(X)$. ***(14 marks)***

End of Question Paper