The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Galois Theory **2 hours 30 minutes plus 30 minutes**

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances.

Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Let $f(x) := x^5 - 3x^3 + 1$.

(i) By reducing modulo 2 or otherwise, show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. You may assume that x^2+x+1 is the only irreducible quadratic polynomial in $\mathbb{F}_2[x]$. ***(4 marks)***

(ii) Show that $f(x)$ has exactly three real roots. ***(4 marks)***

(iii) Is $f(x)$ solvable by radicals over $\mathbb{Q}$? Justify your answer. ***(4 marks)***

2 Let $f(x) := x^4 - 6x^2 + 2 \in \mathbb{Q}[x]$, and let M be the splitting field for f over $\mathbb{Q}$. Set

$$\alpha := \sqrt{3+\sqrt{7}}, \quad \beta := \sqrt{3-\sqrt{7}}$$

where $\sqrt{\ }$ signifies positive square root of positive real numbers.

(i) By considering $\alpha\beta$, or otherwise, show that $M = \mathbb{Q}(\alpha, \sqrt{2})$ and deduce that $[M : \mathbb{Q}]$ is 4 or 8. ***(5 marks)***

You may assume that $[M : \mathbb{Q}] = 8$ *for the rest of this question.*

(ii) Explain how to identify the elements of $\mathrm{Gal}(M/\mathbb{Q})$ by their action on α and $\sqrt{2}$. Calculate the effect of $\mathrm{Gal}(M/\mathbb{Q})$ on β, and use the labelling

$$\alpha_1 := \alpha, \quad \alpha_2 = \beta, \quad \alpha_3 = -\alpha, \quad \alpha_4 := -\beta$$

to derive an identification of $\mathrm{Gal}(M/\mathbb{Q})$ with a subgroup of S_4. The image in S_4 must be written in cycle notation. Finally, compute the action of $\mathrm{Gal}(M/\mathbb{Q})$ on $\alpha+\beta$, $\alpha-\beta$ and $\sqrt{7}$. ***(15 marks)***

Suggestion. You may find it helpful to record the various actions in the form of a table

$\mathrm{Gal}(M/\mathbb{Q})$	$\theta_1 = \mathrm{id}$	...	...
action on α	α	...	...
action on $\sqrt{2}$	$\sqrt{2}$	...	...
action on β	...	...	...
permutations in S_4	...	...	...
action on $\alpha+\beta$	...	...	...
action on $\alpha-\beta$	...	...	...
action on $\sqrt{7}$	...	...	...

(iii) Determine all fields K such that $\mathbb{Q} \subseteq K \subseteq M$ and $[K : \mathbb{Q}] = 4$. ***(8 marks)***

3 Let p be a prime number and let L be the splitting field of the polynomial $x^5 - p$ over $\mathbb{Q}$. Set

$$\alpha := \sqrt[5]{p} \in \mathbb{R} \quad \text{and} \quad \zeta := e^{\frac{2\pi i}{5}}.$$

You are given that $L = \mathbb{Q}(\alpha, \zeta)$, $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$.

(i) Prove that $[L : \mathbb{Q}] = 20$. Deduce that $x^5 - p$ is irreducible over $\mathbb{Q}(\zeta)$. ***(3 marks)***

(ii) Show that if q is a prime number different from p, then $\mathbb{Q}(\sqrt[5]{p}) \neq \mathbb{Q}(\sqrt[5]{q})$. ***(7 marks)***

Hint. For a contradiction, you might want to consider applying an automorphism of L to a possible equation

$$\sqrt[5]{q} = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$$

with $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Q}$.

End of Question Paper