The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Probability with Measure **3 hours**

Candidates should attempt ***ALL*** *questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 40.

This is an open book exam.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.***

Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Let S be a set and let Σ be a σ-field on S. Let X be any subset of S. Show that

$$\Sigma_X = \{A \cap X \; ; \; A \in \Sigma\}$$

is a σ-field on X. ***(4 marks)***

2 Let (S, Σ, m) be a measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets and define

$$A = \bigcap_{n=1}^{\infty} B_n \qquad \text{where} \qquad B_n = \bigcup_{i=n}^{\infty} A_i.$$

(a) Suppose that $\sum_{n=1}^{\infty} m(A_n) < \infty$.

Consider the following ten statements.

A. Since m is a measure, $m(B_n) = \sum_{i=n}^{\infty} m(A_i)$.

B. By a union bound, $m(B_1) \leq \sum_{n=1}^{\infty} m(A_n) < \infty$.

C. By a union bound, $m(B_n) \leq \sum_{i=n}^{\infty} m(A_i) < \infty$.

D. Note that $B_{n+1} \subseteq B_n$ so (B_n) is decreasing.

E. Note that $A_n \subseteq A_{n+1}$ so (A_n) is increasing.

F. Hence $\lim_{n\to\infty} m(B_n) = m(A)$.

G. Hence, for all n, $\lim_{i\to\infty} m(A_i) = m(B_n)$.

H. Hence $m(B_n) \to 0$.

I. Hence A is the empty set.

J. Recall that the Cantor set has measure zero.

Five of the statements **A**-**J** can be used, when arranged into order, to give a proof that $m(A) = 0$. The other five statements are not required.

List which five statements are required. ***(4 marks)***

Hint: The 'union bound' is the inequality $m(\bigcup_{n=1}^{\infty} C_n) \leq \sum_{n=1}^{\infty} m(C_n)$.

(b) Let $c > 0$ and suppose that m is finite. Suppose that $m(A_n) \geq c$ for all n. Show that $m(A) \geq c$. ***(3 marks)***

(c) Give an example in which $m(A_n) \to 0$ and $m(A) > 0$. ***(2 marks)***

3 Explain why the following functions, which map from $\mathbb{R} \to \mathbb{R}$, are Borel measurable. State clearly any standard results that you use in your arguments.

(i) $f(x) = \sin x$

(ii) $g(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ \sin x & \text{if } x < 0 \end{cases}$

(iii) $h(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$

You may assume that $h(x)$ exists for all $x \in \mathbb{R}$.

(4 marks)

4 (a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 6 & \text{if } x \in [0, \frac{1}{3}) \\ 3 & \text{if } x \in [\frac{1}{3}, \frac{1}{2}) \\ 2 & \text{if } x \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(i) Write g and g^2 explicitly as simple functions, where g^2 is defined pointwise: $g^2(x) = g(x)^2$.

(ii) Verify that

$$\int_{\mathbb{R}} g(x)^2 \, dx - \left(\int_{\mathbb{R}} g(x) \, dx \right)^2 = 3.25$$

by computing each of the integrals explicitly.

(4 marks)

(b) Let $C \in (0, \infty)$.

(i) For $i = 1, \ldots, n$ let $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{n} \alpha_i = 1$ and let $c_i \in [0, C]$. Show that

$$\left(\sum_{i=1}^{n} \alpha_i c_i \right)^2 \leq C^2 + \sum_{i=1}^{n} \alpha_i c_i^2.$$

(ii) Let (S, Σ) be a measurable space and let m be a probability measure. Let $f : S \to [0, C]$ be a non-negative measurable function. Show that

$$\left(\int_S f \, dm \right)^2 \leq C^2 + \int_S f^2 \, dm.$$

You may assume a suitable result concerning approximation of non-negative functions with simple functions, providing it is clearly stated. ***(9 marks)***

5 Let $1 < a < b < \infty$. Let $f(x, y) = x^{-y}$, which you may assume is a measurable function $f : [1, \infty) \times [a, b] \to \mathbb{R}$.

Show that f is integrable on $[1, \infty) \times [a, b]$ and hence evaluate

$$\int_1^\infty \frac{x^{-a} - x^{-b}}{\log x}\, dx.$$

(6 marks)

Hint: Note that $f(x, y) = e^{-y \log x}$. *Consider integrating* f *over* $[1, \infty) \times [a, b]$.

6 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of independent events such that $\sum_{n=1}^\infty \mathbb{P}[E_{2n-1}] = \infty$ and $\sum_{n=1}^\infty \mathbb{P}[E_{2n}] < \infty$.

Given this information, which of the following claims are true? Justification is not required.

(i) $\mathbb{P}[E_n] \to 0$ as $n \to \infty$.

(ii) $\mathbb{P}[E_n \text{ infinitely often}] = 1$.

(iii) $\mathbb{P}[E_{2n} \text{ infinitely often}] > 0$.

(iv) $\mathbb{P}[E_{3n} \text{ eventually}] \in \{0, 1\}$.

(v) $\mathbb{P}[E_{3n} \text{ infinitely often}] = 1$.

(vi) Almost surely, (E_n) contains an infinite subsequence of events that occur, and an infinite subsequence of events that do not occur.

(4 marks)

End of Question Paper