The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020-2021**

MAS5051 Probability and Probability Distributions **3 hours 30 mins**

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier.)
Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.
The paper will be marked out of 60. Section A contains fifteen marks and is made up of short questions. Section B contains longer questions, often with related parts. Please be careful to portion your time accordingly.

Section A

A1 A random variable X is distributed according to a Poisson distribution with rate $\lambda = 5$, calculate the probability X takes a value strictly less than five. ***(2 marks)***

A2 Let X_5 be the event a five is rolled on a standard unbiased six sided dice. If the dice is rolled six times, what is the probability we score more than one five? ***(2 marks)***

A3 An urn contains eight red balls and twelve blue balls. Let B be the number of blue balls drawn from the urn and R be the number of red balls drawn. We draw six balls at random without replacment. State an appropriate distribution (with parameters) for this scenario and calculate the probability we draw exactly four red balls.

(2 marks)

A4 A random variable X has probability density function

$$f_X(x) = \begin{cases} 12(x-0.5)^2 & \text{for } x \in [0, 0.5], \\ 1 & \text{for } x \in (0.5, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function of X ***(4 marks)***

A5 Consider two independent random variables X and Y such that $\text{Var}(X) = 4$ and $\text{Var}(Y) = 25$. Let $a = \sqrt{7}$, $b = -4$ and $c = \frac{1}{4}$ be constants. Calculate

$$\text{Var}(aX - bY + c).$$

(2 marks)

A6 A Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{S} = \{1, 2, 3\}$ has transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Calculate the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$. ***(3 marks)***

Section B

B1 (i) Consider the following probability density function for a random variable Y

$$f_Y(y) = \begin{cases} \sum_{i=0}^{\infty} k y^i, & y \in \left(\frac{1}{3}, \frac{1}{2}\right). \\ 0, & \textit{otherwise.} \end{cases}$$

(a) Find k. ***(5 marks)***

(b) Find c such that $\mathbb{P}\left(\frac{5}{12} - c \leq Y \leq \frac{5}{12} + c\right) = \frac{1}{2}$. ***(4 marks)***

(ii) By using the substitution $x = \sin^2(\theta)$, verify that

$$f_X(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}, & x \in (0,1), \\ 0, & \textit{otherwise,} \end{cases}$$

is a valid probability density function for the random variable X. ***(6 marks)***

B2 (i) Consider the multivariate normal distribution given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3 \left(\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & 1 \end{pmatrix} \right).$$

Let $Y_1 = X_1 - X_2 - X_3$ and $Y_2 = X_1 + X_2 + X_3$.

(a) Find the joint distribution of Y_1 and Y_2. ***(5 marks)***

(b) State the covariance between Y_1 and Y_2, $\text{Cov}(Y_1, Y_2)$. ***(1 mark)***

(c) Calculate the correlation between Y_1 and Y_2, $Cor(Y_1, Y_2)$. ***(2 marks)***

(ii) Let $S_k = X_1 + X_2 + \cdots + X_k$ such that X_i for $i \in \{1, 2, \ldots, k\}$ are independently and identically exponentially distributed random variables with rate λ, i.e. $X_i \sim \text{Exp}(\lambda)$. Prove with adequate reasoning that

$$S_k \sim \text{Gamma}(x; k, \lambda).$$

Hint: You may find the moment generating function of a gamma distributed random variable $Z \sim \text{Gamma}(x; k, \lambda)$,

$$M_Z(t) = \left(\frac{1}{1 - \frac{t}{\lambda}} \right)^k, \text{ for } t < \lambda,$$

helpful in your workings. ***(7 marks)***

B3 (i) A Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{S} = \{0,1,2,3\}$ has transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw the transition diagram corresponding to P. ***(1 mark)***

(b) Deduce the general form of P^n in terms of n. ***(4 marks)***

(c) Use your answer from part (b) to calculate $\mathbb{P}(X_{20} = 2 | X_{10} = 2)$ in exact form. ***(2 marks)***

(ii) A player pays £4 to play a game in which they select an ordered pair (a_G, b_G), such that each digit is selected from $\{0,1,2,3,4\}$ with replacement. A random number generator then selects one of the twenty five options each with equal probability. If the player selects at least one digit correctly (in the correct position), they are awarded £W back, such that

$$W = \begin{cases} a_G + b_G, & \text{if the player wins.} \\ 0, & \text{if the player loses.} \end{cases}$$

The profit or loss X is therefore

$$X = -4 + W.$$

(a) Explain with verbal reasoning (i.e. no calculations) why a player would be unwise to choose any of

$$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(3,0)\}.$$

(1 mark)

(b) Calculate the expected winnings of each of the following strategies:

(S1) Selecting $(a_G, b_G) = (4,4)$ every time. ***(2 marks)***

(S2) Selecting (a_G, b_G) at random each with probability $\frac{1}{25}$. ***(5 marks)***

End of Question Paper