The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Financial Mathematics **3 hours**

This is an open book exam.
Answer all three questions. The total mark for this exam is 100.

You can work on the exam during the 24 hour period starting at 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a university-approved calculator). To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (i) Consider a perpetual bond which pays £1 once a year for ever. Assume that the first annual payment is in 6 months, that the bond is traded for £20 and that risk-free spot interest rates for all maturities are identical.

Find the common value of these risk-free spot interest rates. ***(8 marks)***

(ii) Suppose that the price of a stock today is £45. Over the next two 3-months periods (two period binomial tree), it is expected either to increase by 6% or to decrease by 5% (per period). Let us assume that the annual risk-free rate of interest is 6% and that it stays constant. The share pays no dividends. An investment bank came up with a new type of contract that takes into account the whole history of the stock's value, a so-called *Asian call option* with payoff given by

$$A\big(\{S_n\}_{n=0,\cdots,N}\big) = \max\left\{ \left(\frac{1}{N+1} \sum_{n=0}^{N} S_n \right) - X, 0 \right\},$$

with N the total number of periods in the above market model and S_n the price of the underlying asset at time n.

Find the price of the 6-month *Asian call option* with strike $X = £46$ at time $t = 0$. *In your calculations you can round up to two decimal places.* ***(13 marks)***

(iii) Consider a portfolio consisting entirely of European call options on the same stock that pays no dividends, with same expiration time $T > 0$, but with different strike prices. The composition of the portfolio is: long one option with strike 10, long one option with strike 30, long one option with strike 60, as well as one short option with strike 20, and two short options with strike 40.

(a) Derive an explicit formula (as a piecewise-defined function) for the payoff of this portfolio at expiration as a function of the price S_T of the underlying stock. ***(6 marks)***

(b) Sketch this payoff function at expiration. ***(4 marks)***

(c) Consider a second portfolio with only one European put option on the same stock, with expiration at time T and with strike price 60. Compare the two portfolios to deduce, with justification, an inequality between the prices (at time 0) of all the options involved. ***(4 marks)***

2 (i) Let W_t and $\tilde{W}_t$ be two standard independent Brownian Motions and for a constant $-1 \leq \rho \leq 1$, define $X_t := \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$. Is $X := (X_t)_{\{t \geq 0\}}$ a Brownian Motion? ***(8 marks)***

You can assume that the increments of W_t and $\tilde{W}_t$ along the same periods are independent random variables.

(ii) For a fixed $0 < T < \infty$, let $(B_t)_{t\in[0,T]}$ denote the standard Brownian motion. Suppose that the stock price $(S_t)_{t\in[0,T]}$ satisfies the following stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad S_0 = x_0$$

for any $t \in [0,T]$, where r is the risk-free rate of interest, σ is the stock volatility and $x_0 > 0$ is today's stock price.

By using Itô's formula, show that

$$S_t = x_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}, \qquad t \in [0,T].$$

is a solution to the above SDE. ***(10 marks)***

(iii) (a) The spot price for "Amazing Returns Inc" (ARI) stock is £50. Based on historical price data ARI's quants will use Geometric Brownian Motion with expected return 15% and volatility 25% to model the evolution of the stock price in the future. Under this assumption, what is the probability that the stock price will be greater than £70 two years from now? ***(6 marks)***

Calculate to three decimal places. You can leave your final answer in terms of values of the cumulative distribution function of the standard normal.

(b) Consider a derivative on stock of *Fifty-Fifty Ltd.* which entitles its holder to receive a single payment at a future time $T > 0$. The amount of this payment is determined by tossing a fair coin: if the result is heads, the owner of the option receives a payment of twice the *Fifty-Fifty Ltd.* stock price at time T, otherwise the payment is zero.

(α) Find the value of this derivative in a risk-neutral world. ***(3 marks)***

(β) Explain why the value in the real world is not the same as in a risk-neutral world. ***(1 mark)***

(γ) Explain why this discrepancy does not contradict the principle of risk-neutral valuation. ***(2 marks)***

3 (i) Consider two investments A and B whose standard deviations of yearly returns are σ_A and σ_B, respectively. Let ρ be the correlation between the yearly returns of A and B.

(a) Describe the portfolio consisting of £1 invested in combination of holdings in A and B whose return has the lowest standard deviation of returns. ***(8 marks)***

(b) Explain why investors who must invest for a year £1 in portfolios consisting only of A and B will not necessarily invest in the portfolio you found in part (a). ***(2 marks)***

(ii) (a) Explain why the expected returns of the market portfolio must exceed the risk free return. ***(3 marks)***

(b) Show that the expected return of an investment whose beta coefficient is negative must be lower than the risk free return. ***(5 marks)***

(iii) This question is about measures of risk and in particular, the value-at-risk (VaR).

(a) A risk manager is managing a simple portfolio of one asset. She has data for the past 31 trading days, a record of the price of the asset on each of these past 31 days.

Explain how the manager can use historical simulation based on the past returns to estimate the 90% and the 95% daily VaR for potential future losses. Explicitly state any additional assumptions you may want to make. ***(7 marks)***

(b) A colleague of the above risk manager is also looking at the above dataset. He wants to take a different approach to estimate VaR: from the data he found the mean return to be $\mu = 0.04$ and the standard deviation of returns to be $\sigma = 0.32$. For his calculations, he is willing to assume that the distribution of returns is approximately normal. Under this assumption of normality for the returns,

(α) calculate the 90% and 95% daily VaR. ***(8 marks)***

(β) What would these number imply for an investment of £1000 in this asset? ***(2 marks)***

You may use the following values from a table of values for the cumulative distribution function (cdf) Φ of a standard normal random variable: $\Phi(-1.28) = 0.1, \Phi(-1.65) = 0.05$

End of Question Paper