The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020-21**

Fields

This is an open book exam.
*Answer **all** questions.*
*You can work on the exam during the 24 hour period starting at 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period. **Late submission will not be considered without extenuating circumstances.** Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (i) Consider two fields over the field of rational numbers $\mathbb{Q}$,

$$L = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \text{ and } M = \mathbb{Q}((\sqrt{2})^a(\sqrt{3})^c, (\sqrt{2})^b(\sqrt{3})^d)$$

for some integers a, b, c, and d. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Suppose that $\det(A) = 1$. Is it true that $L = M$? Justify your response. ***(5 marks)***

(b) Suppose that $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Is it true that $L = M$? Justify your response. ***(5 marks)***

(ii) Determine the minimal polynomial over $\mathbb{Q}$ of each of the elements $i\sqrt{3}$ and $2^{1/3}$, and show that $[\mathbb{Q}(i\sqrt{3}, 2^{1/3}) : \mathbb{Q}] = 6$. ***(6 marks)***

(iii) Let $a = 2^{2/3}i\sqrt{3}$. Show that $\mathbb{Q}(i\sqrt{3}, 2^{1/3}) = \mathbb{Q}(a)$. ***(5 marks)***

(iv) Find the minimal polynomial $m(x)$ of the element $a = 2^{2/3}i\sqrt{3}$ over $\mathbb{Q}$. ***(4 marks)***

(v) Let $a = 2^{2/3}i\sqrt{3}$. Find $[\mathbb{Q}(a) : \mathbb{Q}(i\sqrt{3})]$ and find the minimal polynomial $m_1(x)$ of the element a over the field $\mathbb{Q}(i\sqrt{3})$. ***(5 marks)***

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2 (i) Let p be a prime number, $\mathbb{F}_p = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$ be the finite field that contains p elements, $\overline{\mathbb{F}_p}$ be its algebraic closure, and

$$f(x) = x^{2p^m} - \left(\left(\sqrt{2}\right)^{p^m} + \left(\sqrt{3}\right)^{p^m}\right)x^{p^m} + \left(\sqrt{6}\right)^{p^m} \in \overline{\mathbb{F}_p}[x]$$

where $m \geq 1$ is a natural number. Let $L \subseteq \overline{\mathbb{F}_p}$ be the splitting field of the polynomial $f(x)$. Find a basis of the field L over the field $\mathbb{F}_p$ and the degree $[L : \mathbb{F}_p]$ in the following cases:

(a) $p = 2$. ***(5 marks)***

(b) $p = 3$. ***(3 marks)***

(c) $p = 5$. ***(3 marks)***

(d) $p = 7$. ***(3 marks)***

(ii) Let p be a prime number, $\{a_n \mid n \geq 0\}$ be the set of elements that are defined recursively as follows: $a_0 = p$, $a_1 = p^2 + p$ and

$$a_n = \Big(\sum_{i=0}^{n-1}(i+2)a_i^{i^2+1} - 1\Big)^n + (-1)^{n+1}\Big(\sum_{i=0}^{n-1}(i+3)a_i^{i^3+1} + 1\Big)^n \quad \text{for} \quad n \geq 2.$$

Which of the polynomials

$$f_n(x) = x^n + \sum_{i=0}^{n-1} a_i x^i, \quad n \geq 2,$$

are irreducible over the field of rational numbers $\mathbb{Q}$. Justify your response. ***(6 marks)***

End of Question Paper