The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2020–2021

MAS6340: Analysis I — 3 hours

This is an open book exam.

Please attempt all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$. We write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1 (i) Let $C_b[1,\infty)$ be the vector space of bounded continuous functions $f\colon [1,\infty) \to \mathbb{K}$. Prove that we have a norm on $C_b[1,\infty)$ defined by the formula

$$\|f\| = \sup\{|f(t)| \mid t \in [1,\infty)\}$$

Prove also that the space $C_b[1,\infty)$ is a Banach space with this norm. ***(12 marks)***

(ii) Let $C[0,1]$ be the normed vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$ with norm $\|f\| = \sup\{f(t) \mid t \in [0,1]\}$. Prove that we have a bounded linear map $S\colon C[0,1] \to \mathbb{R}$ defined by the formula $S(f) = f(0)$. Deduce that the subspace

$$C_0[0,1] = \{f \in C[0,1] \mid f(0) = 0\}$$

of $C_0[0,1]$ is a Banach space; you may use here without proof the facts that the space $C[0,1]$ is a Banach space and the fact that the kernel of a bounded linear map is closed. ***(5 marks)***

(iii) Consider the subspace $C_0[1,\infty) = \{f \in C[1,\infty) \mid f(t) \to 0 \text{ as } t \to \infty\}$ of $C_b[1,\infty)$. Prove that we have a bounded linear map $T\colon C_0[0,1] \to C_0[1,\infty)$ defined by the formula

$$T(f)(t) = f\left(\frac{1}{t}\right) \qquad t \geq 1$$

(4 marks)

(iv) Prove that the map T is invertible, and the inverse of T is a bounded linear map. Use this and the above to prove that the subspace $C_0[1,\infty)$ is a closed subspace of the space $C_b[1,\infty)$. ***(4 marks)***

2 (i) Let $S, T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be linear maps given by the formulae

$$S(f)(x) = \int_{-\infty}^{\infty} e^{-x^2-t^4} f(t)\, dt \qquad T(f)(x) = f(-2x) + f(x-1)$$

respectively.

Prove that S and T are bounded linear maps, and find the adjoints S^* and T^*. ***(12 marks)***

(ii) Let H be a Hilbert space. We call a bounded linear map $P\colon H \to H$ a *projection* if $P = P^* = P^2$. Prove that if P is a projection, then $(im\, P)^\perp = \ker P$ and $im\, P$ is closed. Deduce that

$$H = \ker P \oplus im\, P$$

and P is given by the formula

$$P(v+w) = w \qquad \text{where } v \in \ker P,\ w \in im\, P$$

You may use any standard theorems about Hilbert spaces without proof. ***(8 marks)***

(iii) If P is a projection, prove that $\text{Spectrum}(P) \subseteq \{0,1\}$. You may use the spectral mapping theorem for polynomials without proof. ***(5 marks)***

End of Question Paper