The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Analysis IIA **3 hours**

This is an open book exam.
Answer both questions. The total marks for this exam is 75.

You can work on the exam during the 24 hour period starting at 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 (i) For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, let

$$\rho(x, y) = \max(|x_i - y_i|) + \min(|x_i - y_i|).$$

(a) Show that if $n = 2$ then, for all $x, y \in \mathbb{R}^n$, $\rho(x, y) = d_1(x, y)$. ***(4 marks)***

(b) By considering $(1, 1, 1), (1, 2, 3)$ and $(2, 2, 4)$, show that if $n = 3$ then ρ is not a metric. ***(3 marks)***

(ii) The metrics d_1 and d_∞ on the space $C[0, 1]$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ are given by the rules

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx,$$
$$d_\infty(f, g) = \sup_{0 \le x \le 1} |f(x) - g(x)| \text{ for all } f, g \in C[0, 1].$$

In what follows $f \in C[0, 1]$ is the function such that $f(x) = 0$ for all $x \in [0, 1]$.

(a) Let $a > 0$. Compute $d_1(e^{-ax}, f)$ and $d_1(e^{-ax}, e^{ax})$. ***(4 marks)***

(b) Let $a > 0$ as before. Show that e^{-ax} is strictly decreasing on $[0, 1]$ and hence find $d_\infty(e^{-ax}, f)$. Also find $d_\infty(e^{-ax}, e^{ax})$. ***(6 marks)***

(iii) For $n \geqslant 1$, let $f_n \in C[0, 1]$ be such that $f_n(x) = e^{-nx}$ for all $x \in [0, 1]$.

(a) If $f(x) = 0$ on $[0, 1]$ show that $f_n \to f$ in $(C[0, 1], d_1)$. ***(5 marks)***

(b) Determine whether the function $T : C[0, 1] \to \mathbb{R}$ given by $T(g) = g(0)$ is continuous when $C[0, 1]$ has the metric d_1 and $\mathbb{R}$ has its usual metric. ***(5 marks)***

(iv) Consider $\mathbb{R}^3$ with the Euclidean metric d_2.

(a) Let $((x_n, y_n, 0))$ be a convergent sequence in $\mathbb{R}^3$ with limit (x, y, z). Show that $d_2((x_n, y_n, 0), (x, y, z)) \geqslant |z|$ for all n and deduce that $z = 0$. ***(3 marks)***

(b) Show that $\left(\sin\left(n^{-\frac{3}{2}}\right), \ln\left(\frac{n+1}{n}\right)\right) \to (0, 0)$ in the metric space $(\mathbb{R}^2, d_2)$. ***(4 marks)***

2 In this question d_1 and d_2 stand for the usual (taxicab and Euclidean respectively) metrics in $\mathbb{R}^2$.

Define
$B_2 := \{x \in \mathbb{R}^2 : d_2(x, (4, 0)) \leq 2\}$
$E_2 := \{x \in \mathbb{R}^2 : d_2(x, (0, 0)) \leq 1\}$, and

For this question $(X, d) = (\mathbb{R}^2, d_2)$.

(i) Let $\{z_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence, with respect to d_2, in $E_2 \cup B_2$. Show that there is $N \in \mathbb{N}$ such that $\{z_n\}$ is fully contained in one of E_2 or B_2 for all $n \geq N$. ***(7 marks)***

Hint: first prove that $\min_{\mathbf{x}\in E_2, \mathbf{y}\in B_2} d_2(\mathbf{x}, \mathbf{y}) = 1$.

(ii) Use the above to show that $E_2 \cup B_2$ is complete in (X, d). ***(7 marks)***

Note: Arguments not using Cauchy sequences explicitly will only get partial credit.

(iii) Show that $E_2^c \cap B_2^c$ is open, where D^c stands for the complement of a set D. ***(4 marks)***

(iv) Prove that $E_2 \cup B_2$ can be fully covered by a finite number of open balls, all with radius $r = \dfrac{3\pi}{17\sqrt{2}}$. ***(5 marks)***

(v) Define the sequence $\vec{x}_n = (a_n, b_n)$, where $a_n \in [2, 6], b_n \in [-2, 2]$, for all $n \in \mathbb{N}$.

(a) Show that $\vec{x}_n$ has at least one subsequence that converges in (X, d). ***(6 marks)***

(b) Assuming that $\{a_n\}_n, \{b_n\}_n$ are Cauchy sequences in $\mathbb{R}$ (with its usual metric), prove that all subsequences of $\vec{x}_n$ converge to the limit of the subsequence considered in the previous part. ***(6 marks)***

(vi) Consider the recurrence $x_{n+1} = 2 + \dfrac{1}{2}\sin(x_n), x_0 = 2$.
Use the Contraction Mapping Principle to prove that x_n has a limit L.
You need to fully justify your answer. ***(6 marks)***

End of Question Paper