The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2020–2021

Topics in Advanced Fluid Mechanics — **3 hours**

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Consider a model equation for the vorticity ω in $\mathbb{R}^1$

$$\frac{\partial \omega}{\partial t} = \omega H[\omega]. \qquad (1)$$

Here $H[\omega]$ denotes the Hilbert transform $H[\omega](x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy$, defined with a principal-value integral. We consider a solution of the form

$$\omega(x,t) = \frac{x}{x^2 + a(t)^2},$$

where $a(t) > 0$ for $t \geq 0$. You may assume the following formulae

$$H\left[\frac{a}{x^2+a^2}\right] = \frac{x}{x^2+a^2}, \; H\left[\frac{x}{x^2+a^2}\right] = -\frac{a}{x^2+a^2}.$$

The Fourier transform $\tilde{f}(k) \equiv \mathcal{F}(f)(k)$ of a function $f(x)$ is defined by

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dk.$$

(i) Explain briefly why $\tilde{\omega}(k,t)$ satisfies the following equation:

$$\frac{\partial \tilde{\omega}(k,t)}{\partial t} = \tilde{\omega} * \widetilde{H[\omega]}(k,t) \; ,$$

where $*$ indicates convolution. ***(4 marks)***

(ii) Given that $\mathcal{F}\left(\frac{a}{x^2+a^2}\right) = \frac{1}{2}\exp(-a|k|)$, $(a > 0)$, find $\tilde{\omega}(k,t)$ and hence show that

$$\frac{\partial \tilde{\omega}(k,t)}{\partial t} = \frac{i}{2} k \frac{da(t)}{dt} e^{-a|k|}$$

(7 marks)

(iii) Show that

$$\mathcal{F}(H[\omega])(k,t) = -\frac{1}{2}\exp(-a|k|) \; .$$

(4 marks)

(iv) By evaluating the convolution, show that the right-hand side of the Fourier-transformed equation in part (i) is equal to

$$\frac{i}{4} k e^{-a|k|}.$$

Hint: consider $k \geq 0$ and $k < 0$ separately. ***(23 marks)***

(v) Determine $a(t)$ explicitly. Describe qualitatively how the regularity property of the solution $\omega(x,t)$ changes as t increases. Justify your answer briefly.

(12 marks)

2 (i) Consider the impulse formulation for the 3D incompressible Euler equations where $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$,

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma}$$

$$\frac{D\phi}{Dt} = p - \frac{|\boldsymbol{u}|^2}{2},$$

where $\nabla \cdot \boldsymbol{u} = 0$.

(a) Show that $\frac{D}{Dt}\boldsymbol{\gamma} \cdot \boldsymbol{\omega} = 0$. ***(7 marks)***

(b) Derive an expression for $\frac{D}{Dt}\boldsymbol{u} \cdot \boldsymbol{\omega}$ of the form $\boldsymbol{\omega} \cdot \nabla A$, where A is a function of p and $\boldsymbol{u}$ to be found. ***(8 marks)***

(c) Assume $\boldsymbol{\gamma}(\boldsymbol{a}, t) \cdot \boldsymbol{\omega}(\boldsymbol{a}, t) \neq 0$ for a fluid particle $\boldsymbol{a}$. If $|\boldsymbol{\gamma}(\boldsymbol{a}, t)| \to \infty$ as $t \to \infty$, briefly comment on the geometrical features of the flow field at $\boldsymbol{a}$. ***(5 marks)***

(d) Compute $\boldsymbol{\gamma} \cdot \boldsymbol{\omega}$ for the case of two-dimensional flow. ***(5 marks)***

(ii) Consider the velocity field

$$\boldsymbol{u} = (-ax, v(x, t), az)$$

in Cartesian coordinates, governed by the Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u},$$

and the continuity equation

$$\nabla \cdot \boldsymbol{u} = 0,$$

where $a(> 0)$ is a constant (independent of both time and space). We assume that no fields depend on y.

(a) Write down the x and z-components of the Navier-Stokes equation. ***(4 marks)***

(b) Determine the pressure p explicitly. ***(6 marks)***

(c) Write down the y-component of the Navier-Stokes equation. ***(2 marks)***

(d) Show that the vorticity takes the form $\boldsymbol{\omega} = (0, 0, \omega(x, t))$, where $\omega = \partial_x v$. Derive the equation for $\omega(x, t)$. ***(6 marks)***

(e) Find the steady solution $\omega(x)$. ***(7 marks)***

End of Question Paper