The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Mathematical Methods And Modelling Of Natural Systems

This is an open book exam.

Answer **both** *questions. The marks awarded to each section of question are shown in italics. The total mark for the paper is 50.*

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours 30 min of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 The Laplace transform $\tilde{x}(s)$ of $x(t)$ is defined by

$$\tilde{x}(s) = \mathcal{L}\{x(t)\}(s) = \int_0^\infty e^{-st}\, x(t)\, dt.$$

You may assume that Re s is large enough for all Laplace transforms in this question to converge.

(a) For $t > 0$, by direct integration show that

$$\mathcal{L}\left\{\int_0^t x(u)\, du\right\}(s) = \frac{\tilde{x}(s)}{s}.$$ ***(5 marks)***

(b) For $t > 0$, $x(t)$ satisfies the equation

$$\dddot{x} + 8\dot{x} + 16\int_0^t x(u)\, du = f(t) \qquad (1)$$

for a general function $f(t)$, where $\dot{x} = dx/dt$ etc., with $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$.

Use Laplace transforms to find $x(t)$ in terms of f. ***(10 marks)***

(c) Using the answer from (b), show that if $f(t) = t$ then

$$x(t) = \frac{1}{16}(1 - \cos 2t - t\sin 2t).$$ ***(5 marks)***

Verify that this satisfies (1). ***(5 marks)***

2 Consider the equation

$$\epsilon x^3 + 3x^2 + 5x - 2 = 0, \qquad (2)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

The solution to equation (2) can be written as

$$x = \frac{1}{\epsilon}(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots)$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to find the three solutions of (2), in each case correct to order ϵ as $\epsilon \to 0$. ***(25 marks)***

End of Question Paper