The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Waves and Magnetohydrodynamics A

This is an open book exam.
Answer ***all*** *questions.*
You can work on the exam during the 24 hour period starting at 10am (BST), and you must submit your work within 2 hrs and 30 min of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.*
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Use D'Alembert's solution to find the solution $y(x,t)$ to the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

subject to the initial conditions

$$y(x,0) = 0, \qquad \frac{\partial y}{\partial t}(x,0) = V\cos(kx),$$

where V and k are constants. Find $y(x,t)$ and all values of x for which $y(x,t) = 0$. Describe the motion very briefly.

(10 marks)

2 Calculate the Fourier series of the periodic function $f(x)$ with fundamental period $T = 2\pi$ defined on $(-\pi, \pi]$ by

$$f(x) = \begin{cases} 0 & -\pi < x < 0, \\ 1 & 0 \le x \le \pi. \end{cases}$$

(10 marks)

3 A uniform finite string of length l and mass per unit length ρ occupies the interval $0 \le x \le l$ and undergoes transverse vibrations with displacement $y(x,t)$, where $c^2 y_{xx} = y_{tt}$, and c^2 is a constant. You are given that
(i) $y(0,t) = y(l,t) = \dot{y}(x,0) = 0$;
(ii) $y(x,0) = f(x)$, where $f(x)$ is a known function;
(iii) $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$, with a_n $(n = 1, 2, 3, ..)$ constants.
Derive an expression for the potential energy stored in the string at time t.

(10 marks)

4 The equilibrium position of the free surface of an inviscid incompressible and irrotational *infinite* liquid is $z = 0$, where z is measured vertically upwards. A linear surface wave causes the displacement of this surface to be $\eta(x,t)$, where x is measured along the free surface and

$$\eta = a\sin(kx - \omega t),$$

where a, k and ω are positive constants. You are given that the velocity potential $\phi(x,z,t)$ satisfies the relevant boundary conditions and the equation modelling the incompressible and irrotational nature of the liquid. Use this modelling equation and the appropriate boundary conditions to (i) find $\phi(x,z,t)$ and (ii) derive the accompanying dispersion relation $\omega(k)$.

(10 marks)

5 Using the method of characteristics solve the equation

$$z_x - y^3 z_y = 0,$$

with $z = 1/(1+y^2)$ on $x = 0$, $-\infty < y < \infty$. Explain why the solution is not defined when $x \geq 1/(2y^2)$. $\left[\textit{We use the notation } z_x = \dfrac{\partial z}{\partial x} \textit{ etc.} \right]$

(10 marks)

End of Question Paper