SCHOOL OF MATHEMATICS AND STATISTICS  Autumn Semester 2007-2008
Mathematics III (Electrical)  2 hours

Attempt ALL questions.

1  z is the complex number x + jy.
   (i) Sketch the regions in the z-plane corresponding to x ≥ 3, y ≤ (2x + 1),
       |z| ≤ 3 and |z + 1 − 2j| ≥ 1.  (8 marks)
   (ii) The mapping w = az + b, where a and b are complex numbers, maps the
         point z = 1 to the point w = 1 − j and the point z = 1 − j to the point
         w = 3 − 4j.
          (a) Show that a = 3 + 2j and b = −2 − 3j.
          (b) Using these values of a and b and given that w = u + jv, find
               u(x, y) and v(x, y).
          (c) Find the image in the w-plane of the region |z| ≥ 3 in the z-plane
               under this mapping.  (12 marks)
   (iii) Find out if the image of straight line z = (1 + j)t + 3, where t is real, under
         bilinear mapping w = \frac{z + j}{z - j} is a circle or a straight line.  (5 marks)

2  Expand the function f(z) = \frac{z + 1}{z^2 - 4} into partial fractions and hence
   (i) using either the binomial expansion or the Taylor series method, find the
       first two non-zero terms of the power series expansion of the function f(z)
       about the point z = −j. Indicate the region of convergence and all poles
       and zeros of the function on the Argand diagram.  (17 marks)
   (ii) find the first four terms of the Laurent series expansion of the function
        f(z) about z = 2. Determine and sketch the region of validity of the
        series.  (8 marks)
3 (i) Find all the poles of \( f(z) = \frac{(z + 1)^3}{z^2(z^2 + 2z + 2)} \) and plot them on an Argand diagram. Hence evaluate the integral \( \int_C f(z)dz \), writing your solutions in the form \( a + jb \), where \( a \) and \( b \) are real, where

(a) \( C \) is the circle \( |z| = 4 \)

(b) \( C \) is the circle \( |z - 5| = 2 \).

\( (15 \text{ marks}) \)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

\[
I = \int_{-\infty}^{\infty} \frac{2}{2 + 2x + x^2} \, dx
\]

\( (10 \text{ marks}) \)

4 (i) The function \( y(t) \) satisfies the differential equation

\[
y' - 4y + 3y = \delta(t - 1)
\]

(where dot denotes differentiation with respect to \( t \)) and the initial conditions \( y(0) = 3 \), \( \dot{y}(0) = 9 \). Show that the Laplace transform \( Y(s) \) of \( y(t) \) is given by

\[
Y(s) = \frac{1}{s - 3} \left( 3 + \frac{e^{-s}}{s - 1} \right)
\]

Hence determine \( y(t) \), for \( t > 0 \).

\( (15 \text{ marks}) \)

(ii) Sketch the two-sided exponential pulse given by:

\[
f(t) = \begin{cases} 
\exp(2t), & t \leq 0, \\ 
\exp(-t/2), & t > 0.
\end{cases}
\]

Using direct integration, show that the Fourier transform, \( F(\omega) \), of \( f(t) \) is

\[
\frac{5(2 - 3j\omega + 2\omega^2)}{(1 + 4\omega^2)(4 + \omega^2)}
\]

Sketch the function \( g(t) \) which has Fourier transform

\[
G(\omega) = \exp(j\omega)F(\omega)
\]

\( (10 \text{ marks}) \)

End of Question Paper
### FORMULA SHEET

#### Table of Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>Region of validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant $= c$</td>
<td>$\frac{c}{s}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$e^{\alpha t}$</td>
<td>$\frac{1}{s-\alpha}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$\cos kt$</td>
<td>$\frac{s}{s^2+k^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$\sin kt$</td>
<td>$\frac{k}{s^2+k^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$t^n e^{\alpha t}$</td>
<td>$\frac{n!}{(s-\alpha)^{n+1}}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$e^{\alpha t} \sin kt$</td>
<td>$\frac{k}{(s-\alpha)^2+k^2}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$e^{\alpha t} \cos kt$</td>
<td>$\frac{s-\alpha}{(s-\alpha)^2+k^2}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$\delta(t-T)$</td>
<td>$e^{-sT}$</td>
<td>delta function</td>
</tr>
<tr>
<td>$H(t-T)$</td>
<td>$\frac{e^{-sT}}{s}$</td>
<td>step function</td>
</tr>
<tr>
<td>$H(t) - H(t-T)$</td>
<td>$\frac{1}{s}(1-e^{-sT})$</td>
<td>rectangular pulse</td>
</tr>
</tbody>
</table>

**Note:** in this table the parameters $\alpha$ and $k$ are real constants and $H$ is the Heaviside step function.
Some general properties of the Laplace transform

In the following table the notation $L\{f(t)\} = F(s)$ has been used.

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$L{af(t) + bg(t)} = aL{f(t)} + bL{g(t)}$</td>
</tr>
<tr>
<td>Differentiation w.r.t. $t$</td>
<td>$L{\frac{d}{dt}f(t)} = sF(s) - f(0)$</td>
</tr>
<tr>
<td>Differentiation twice with respect to $t$</td>
<td>$L{\frac{d^2}{dt^2}f(t)} = s^2F(s) - sf(0) - f'(0)$</td>
</tr>
<tr>
<td>Integration</td>
<td>If $g(t) = \int_0^t f(u)du$ then $L{g(t)} = \frac{1}{s}F(s)$</td>
</tr>
<tr>
<td>Differentiation w.r.t. $s$</td>
<td>$L{tf(t)} = -\frac{dF}{ds}$</td>
</tr>
<tr>
<td>Shift</td>
<td>$L{e^{-kt}f(t)} = F(k + s)$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$L{f(at)} = \frac{1}{</td>
</tr>
<tr>
<td>Time delay</td>
<td>$L{f(t-a)H(t-a)} = e^{-as}F(s)$</td>
</tr>
</tbody>
</table>

**Convolution**

For causal functions

$$f \ast g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{0}^{t} f(\tau)g(t-\tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

**Fourier transform**

The Fourier transform $F(\omega)$ of a function $f(t)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt.$$  

The time shift property: the Fourier transform of a function $f(t-T) = e^{-j\omega T}F(\omega)$.

The scaling property: the Fourier transform of a function $f(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$.

**Residues**

The general formula for the residue at a pole, $z_0$, of order $m$ is

$$\lim_{z \to z_0} \frac{1}{(m-1)!} \left( \frac{d^{m-1}}{dz^{m-1}} \left( (z - z_0)^m f(z) \right) \right).$$