Data Provided: Formula Sheet

AMA242

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2007-2008

Mathematics III(Electrical) 2 hours

*Attempt **ALL** questions.*

1 z is the complex number $x + jy$.

(i) Sketch the regions in the z-plane corresponding to $x \geq 3$, $y \leq (2x+1)$, $|z| \leq 3$ and $|z + 1 - 2j| \geq 1$. ***(8 marks)***

(ii) The mapping $w = az + b$, where a and b are complex numbers, maps the point $z = 1$ to the point $w = 1 - j$ and the point $z = 1 - j$ to the point $w = 3 - 4j$.

(a) Show that $a = 3 + 2j$ and $b = -2 - 3j$.

(b) Using these values of a and b and given that $w = u + jv$, find $u(x, y)$ and $v(x, y)$.

(c) Find the image in the w-plane of the region $|z| \geq 3$ in the z-plane under this mapping. ***(12 marks)***

(iii) Find out if the image of straight line $z = (1+j)t+3$, where t is real, under bilinear mapping $w = \dfrac{z+j}{z-j}$ is a circle or a straight line. ***(5 marks)***

2 Expand the function $f(z) = \dfrac{z+1}{z^2-4}$ into partial fractions and hence

(i) using either the binomial expansion or the Taylor series method, find the first two non-zero terms of the power series expansion of the function $f(z)$ about the point $z = -j$. Indicate the region of convergence and all poles and zeros of the function on the Argand diagram. ***(17 marks)***

(ii) find the first four terms of the Laurent series expansion of the function $f(z)$ about $z = 2$. Determine and sketch the region of validity of the series. ***(8 marks)***

3 (i) Find all the poles of $f(z) = \dfrac{(z+1)^3}{z^2(z^2+2z+2)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, where

(a) C is the circle $|z| = 4$

(b) C is the circle $|z - 5| = 2$.

(15 marks)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{2}{2 + 2x + x^2}\, dx$$

(10 marks)

4 (i) The function $y(t)$ satisfies the differential equation

$$\ddot{y} - 4\dot{y} + 3y = \delta(t-1)$$

(where dot denotes differentiation with respect to t) and the initial conditions $y(0) = 3$, $\dot{y}(0) = 9$. Show that the Laplace transform $Y(s)$ of $y(t)$ is given by

$$Y(s) = \frac{1}{s-3}\left(3 + \frac{e^{-s}}{s-1}\right)$$

Hence determine $y(t)$, for $t > 0$. ***(15 marks)***

(ii) Sketch the two-sided exponential pulse given by:

$$f(t) = \begin{cases} \exp(2t), & t \le 0, \\ \exp(-t/2), & t > 0. \end{cases}$$

Using direct integration, show that the Fourier transform, $F(\omega)$, of $f(t)$ is

$$\frac{5(2 - 3j\omega + 2\omega^2)}{(1 + 4\omega^2)(4 + \omega^2)}$$

Sketch the function $g(t)$ which has Fourier transform

$$G(\omega) = \exp(j\omega)F(\omega)$$

(10 marks)

End of Question Paper

FORMULA SHEET

Table of Laplace Transforms

$f(t)$	$F(s)$	Region of validity
constant $= c$	$\frac{c}{s}$	$Re(s) > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$Re(s) > \alpha$
t	$\frac{1}{s^2}$	$Re(s) > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
$e^{\alpha t} \sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$e^{\alpha t} \cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$\delta(t-T)$	e^{-sT}	delta function
$H(t-T)$	$\frac{e^{-sT}}{s}$	step function
$H(t) - H(t-T)$	$\frac{1}{s}(1 - e^{-sT})$	rectangular pulse

Note: in this table the parameters α and k are real constants and H is the Heaviside step function.

Some general properties of the Laplace transform

In the following table the notation $\mathbf{L}\{f(t)\} = F(s)$ has been used.

$\mathbf{L}\{af(t) + bg(t)\} = a\mathbf{L}\{f(t)\} + b\mathbf{L}\{g(t)\}$	linearity
$\mathbf{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathbf{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0) - f'(0)$	differentiation twice with respect to t
If $g(t) = \int_0^t f(u)du$ then $\mathbf{L}\{g(t)\} = \frac{1}{s}F(s)$	integration
$\mathbf{L}\{tf(t)\} = -\frac{dF}{ds}$	differentiation w.r.t. s
$\mathbf{L}\{e^{-kt}f(t)\} = F(k+s)$	shift
$\mathbf{L}\{f(at)\} = \frac{1}{\|a\|}F\left(\frac{s}{a}\right)$	scaling
$\mathbf{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$	time delay

Convolution

For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Fourier transform

The Fourier transform $\mathbf{F}(\omega)$ of a function $f(t)$ is defined by

$$\mathbf{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt.$$

The time shift property: the Fourier transform of a function $f(t-T) = e^{-j\omega T}\ \mathbf{F}(\omega)$.

The scaling property: the Fourier transform of a function $f(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$.

Residues

The general formula for the residue at a pole, z_0, of order m is

$$\frac{1}{(m-1)!}lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right).$$