1. (i) The function $F(x, y)$ is defined by

$$F(x, y) = x^2 + kxy + y^2,$$

where $k$ is a constant and $k^2 \neq 4$.

Show that $F(x, y)$ has just one stationary point and find this stationary point.

Determine the nature of the stationary point, considering separately the cases (i) $k^2 > 4$ and (ii) $k^2 < 4$.

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = (x - 2)^2 + (y - 4)^2 + (z - 4)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 1.$$
2  (i) Evaluate the double integral

\[ I_1 = \int_0^1 \int_0^x (3 - x - y) \, dy \, dx. \]  

(5 marks)

Sketch the region over which the integral \( I_1 \) is defined.  

(3 marks)

(ii) Evaluate \( I_1 \) by changing the order of integration.  

(9 marks)

(iii) Use polar co-ordinates to evaluate the integral

\[ I_2 = \iint_R e^{(x^2+y^2)} \, dx \, dy \]

where \( R \) is the semi-circular disk \( x^2 + y^2 \leq 4, \, y > 0 \).  

(8 marks)

3  Two scalar fields \( U(x, y, z) \) and \( V(x, y, z) \) are given by

\[ U(x, y, z) = 3x^2y, \quad V(x, y, z) = xz^2 - 2y. \]

(i) Find \( \nabla U, \nabla V, \nabla^2 U \) and \( \nabla^2 V \).  

(10 marks)

(ii) Find a unit vector in the direction of the vector \( \mathbf{n} = (0, 5, 12) \).  
      Hence find the directional derivatives of \( U \) and \( V \) in the direction of the vector \( \mathbf{n} \).  

(5 marks)

(iii) A vector field \( \mathbf{G} \) is defined by

\[ \mathbf{G} = (\nabla U) \times (\nabla V). \]

Show that \( \mathbf{G} \) is given by

\[ \mathbf{G} = 6x^3zi - 12x^2yzj + (-12xy - 3x^2z^2)k. \]

Find \( \text{div} \, \mathbf{G} \) and \( \text{curl} \, \mathbf{G} \).  

(10 marks)
4 A vector field $B$ is given by

$$B = (5xy - 6x^2) i + (2y - 4x) j.$$ 

Calculate the line integral

$$\int_C B \cdot dr$$

for the following curves $C$:

(i) The straight line from the origin to the point $(0, 8)$, followed by the straight line from the point $(0, 8)$ to the point $(2, 8)$. \hspace{1cm} (11 marks)

(ii) The straight line from the origin to the point $(2, 8)$. \hspace{1cm} (7 marks)

(iii) The curve $y = x^3$ from the origin to the point $(2, 8)$. \hspace{1cm} (7 marks)

End of Question Paper
Formula Sheet for AMA243 Trigonometry

\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B
\]
\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]
\[
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]

\[
a \cos \theta + b \sin \theta = R \cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos \alpha = \frac{a}{R}, \sin \alpha = \frac{b}{R}
\]

\[
\cos^2 \theta = \frac{1}{2} (\cos 2\theta + 1)
\]
\[
\cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos 3\theta)
\]
\[
\cos^4 \theta = \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta)
\]
\[
\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)
\]
\[
\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta)
\]
\[
\sin^4 \theta = \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta)
\]