SCHOOL OF MATHEMATICS AND STATISTICS  
Spring Semester 2008–2009

Mathematics IV (Electrical)

Attempt all FOUR questions.

1. (i) Find and classify the stationary points of the function
   \[ F(x, y) = x^2 + y^2 + 2x^2y - 3y. \]
   (11 marks)

   (ii) Use the method of Lagrange multipliers to find the maximum and minimum values attained by the function
   \[ f(x, y, z) = (x + 1)^2 + (2y + 1)^2 + (3z + 1)^2 \]
   subject to the constraint
   \[ x^2 + 4y^2 + 9z^2 = 12. \]
   (14 marks)

2. (i) Sketch the region over which the integral \( I \) is defined, where
   \[ I = \int_0^1 \int_y^1 4x^6e^{x^3}y \, dx \, dy. \]
   (3 marks)

   Evaluate \( I \) by changing the order of integration.
   (8 marks)

   (ii) Let \( R \) be the region consisting of all points \((x, y)\) such that \(x \geq 0, y \geq 0\) and \(1 \leq x^2 + y^2 \leq 4\). For which values of the polar co-ordinates \(r\) and \(\theta\) is the point \((r, \theta)\) in the region \(R\)?
   (3 marks)

   Evaluate the integral
   \[ \iint_R (1 + x)\sqrt{x^2 + y^2} \, dA. \]
   (11 marks)
3 A vector field \( \mathbf{A} = \mathbf{A}(x, y, z) \) is given by

\[
\mathbf{A} = (8xz - z^2) \mathbf{i} + (3y^2) \mathbf{j} + (ax^2 - 2xz) \mathbf{k}
\]

for some scalar \( a \).

Calculate \( \text{div} \mathbf{A} \) and \( \text{curl} \mathbf{A} \).  \( (5 \text{ marks}) \)

By evaluating both sides, verify that

\[
\nabla^2 \mathbf{A} = \text{grad div} \mathbf{A} - \text{curl curl} \mathbf{A}.
\]

\( (10 \text{ marks}) \)

Find the value of \( a \) for which \( \text{curl} \mathbf{A} = 0 \) and find a scalar field \( \phi = \phi(x, y, z) \) such that, for this value of \( a \),

\[
\mathbf{A} = \text{grad} \phi.
\]

\( (10 \text{ marks}) \)

4 Let \( S \) be the surface consisting of the hemisphere \( S_1 \) given, in spherical co-ordinates, by \( r = 1, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \frac{\pi}{2} \) together with the disc \( S_2 \) given by \( 0 \leq r \leq 1, 0 \leq \phi \leq 2\pi, \theta = \frac{\pi}{2} \). Let \( V \) be the hemispherical volume enclosed by \( S \) and let \( \mathbf{E} \) be the vector field

\[
\mathbf{E} = xi + yj + (1 - z)k.
\]

(i) Evaluate the surface integral

\[
\iint_S \mathbf{E}.dS. \quad (15 \text{ marks})
\]

(ii) Evaluate

\[
\iiint_V \text{div} \mathbf{E} \, dV
\]

and verify that

\[
\iiint_V \text{div} \mathbf{E} \, dV = \iint_S \mathbf{E}.dS. \quad (10 \text{ marks})
\]

End of Question Paper
Formula Sheet for AMA243 Trigonometry

\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B
\]

\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]

\[
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]

\[a \cos \theta + b \sin \theta = R \cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos \alpha = \frac{a}{R}, \sin \alpha = \frac{b}{R}\]

\[
\cos^2 \theta = \frac{1}{2} (\cos 2\theta + 1)
\]

\[
\cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos 3\theta)
\]

\[
\cos^4 \theta = \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta)
\]

\[
\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)
\]

\[
\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta)
\]

\[
\sin^4 \theta = \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta)
\]