Rings and Groups

Answer Question 1 and three other questions. You are advised not to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.

1  

(i) Use Euclid’s algorithm to find the inverse of 25 in the ring $\mathbb{Z}_{47}$. \( \text{(6 marks)} \)

(ii) Is each of the following rings an integral domain? Justify your answer briefly.

(a) $\mathbb{C}$
(b) $\mathbb{Z}_6$
(c) $\mathbb{Z}_{11}$
(d) $\mathbb{Z}_{11}[x]$
(e) $\mathbb{Z}[\sqrt{-5}]$ \( \text{(8 marks)} \)

(iii) Write down all possible cycle types in $S_4$, together with the number of elements in $S_4$ of each type. Hence write down the class equation for $S_4$. Justify your answers. \( \text{(11 marks)} \)
2  (i) (a) What are the units in the ring \( \mathbb{Z}_{10} \)? Justify your answer, and for each unit give its multiplicative inverse.  
\[(7 \text{ marks})\]

(b) Show that the group of units of \( \mathbb{Z}_{10} \) is cyclic.  
\[(4 \text{ marks})\]

(ii) Let \( d \) be a square-free integer with \( d \neq 1 \). Recall that the norm of an element \( r = a + b\sqrt{d} \) of \( \mathbb{Z}[(\sqrt{d})] \), where \( a, b \in \mathbb{Z} \), is given by
\[N(a + b\sqrt{d}) = |a^2 - b^2d| \]

(a) Show that \( \mathbb{Z}[(\sqrt{-7})] \) has no element of norm 2. Hence show that any element of norm 4 or 8 is irreducible.  
\[(8 \text{ marks})\]

(b) Write down an element of norm 8 in \( \mathbb{Z}[(\sqrt{-7})] \). Hence express 8 as a product of irreducible factors in \( \mathbb{Z}[(\sqrt{-7})] \) in two different ways, and deduce that \( \mathbb{Z}[(\sqrt{-7})] \) is not a unique factorisation domain. Justify your answer.  
\[(6 \text{ marks})\]

3  (i) Are the following elements zero-divisors in the given rings? Justify your answers.

(a) \( 3 \in \mathbb{Z} \)

(b) \( 3 \in \mathbb{R} \)

(c) \( 3 \in \mathbb{Z}_n \) where \( n = 123123123123123123123123 \)

(d) \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \)  
\[(6 \text{ marks})\]

(ii) (a) Let \( R \) be an integral domain. Prove that the units in the polynomial ring \( R[x] \) are precisely the units in \( R \).  
\[(8 \text{ marks})\]

(b) Find a non-constant polynomial in \( \mathbb{Z}_9[x] \) which is a unit, justifying your answer.  
\[(3 \text{ marks})\]

(iii) Calculate the norm of \( 4 + i \in \mathbb{Z}[i] \). Hence exhibit a prime number that is not irreducible in \( \mathbb{Z}[i] \). Does 13 have this property? Justify your answers carefully.  
\[(8 \text{ marks})\]
4  (i) Let $G$ be a group of order 10 with trivial centre.
   (a) Find the class equation for $G$, justifying your answer. \( (5 \text{ marks}) \)
   (b) Find the number of elements of order 5 in $G$. \( (6 \text{ marks}) \)
   (c) Let $h \in G$ be an element of order 5. Let $H = \langle h \rangle$, the subgroup generated by $h$. Use the class equation to show that $H$ is a normal subgroup of $G$. \( (4 \text{ marks}) \)

(ii)  (a) Let $B$ be a normal subgroup of a group $A$. Describe the quotient group $A/B$. (You need to specify what the elements are, how multiplication on $A/B$ is defined, what the identity is, and what the inverse of a given element is, but you do not need to prove any of your assertions.) \( (5 \text{ marks}) \)

   (b) Now let $A$ be the cyclic group of order 4 with elements $1, a, a^2, a^3$. Let $B$ be the subgroup generated by the element $a^2$. Show that the quotient group $A/B$ is a cyclic group of order 2. \( (5 \text{ marks}) \)

5  (i) Let $G$ be a group of order 7.
   (a) Prove that $G$ must be cyclic. \( (4 \text{ marks}) \)
   (b) Hence find the class equation for $G$, justifying your answer carefully. \( (6 \text{ marks}) \)

(ii)  (a) State, without proof, the First Isomorphism Theorem for groups. \( (2 \text{ marks}) \)

   (b) Consider a square with vertices labelled as shown.

   \[
   \begin{array}{cc}
   1 & 2 \\
   4 & 3 \\
   \end{array}
   \]

   Write $D_4$ for the group of symmetries of the square. Explain briefly how the action of $D_4$ on the vertices of the square gives rise to a homomorphism $f : D_4 \rightarrow S_4$. \( (4 \text{ marks}) \)

   (c) Find the kernel and image of the above homomorphism $f$, justifying your answer. \( (7 \text{ marks}) \)

   (d) What does the First Isomorphism Theorem tell us in this case? \( (2 \text{ marks}) \)

End of Question Paper