SCHOOL OF MATHEMATICS AND STATISTICS

Mathematics III (Electrical)

2 hours

Attempt ALL questions.

1. For the function \( w = 1/z \) find the image of the following in the \( w \)-plane. Sketch your results in the \( z \) and \( w \)-planes. For (ii) and (iii) \( 0 \leq r < \infty \), and for (iv) \( 0 \leq \theta < 2\pi \).
   (i) The point \( z = 1 - j \). (3 marks)
   (ii) The line \( w = re^{-j\theta/2} \). (5 marks)
   (iii) The line \( z = -1 + re^{j\theta/2} \). (13 marks)
   (iv) The circle \( z = 2e^{j\theta} \). (4 marks)

2. (i) Find the expansion in the Taylor series of the function \( \frac{1}{1+z} \) about \( z = 2j \). Show the region of convergence on the Argand diagram and indicate the pole that determines the radius of convergence. (10 marks)

   (ii) Find the Laurent series expansion of \( \frac{1}{2j + (2-j)z - z^2} \) in the region \( 1 < |z| < 2 \). (15 marks)
3 (i) Find all the poles of \( f(z) = \frac{z^4}{z^4 + 13z^2 + 36} \) and plot them on an Argand diagram. Hence evaluate the integral \( \int_C f(z) dz \), writing your solutions in the form \( a + jb \), where \( a \) and \( b \) are real, where

(a) \( C \) is the circle \( |z| = 4 \)

(b) \( C \) is the circle \( |z + 2j| = 2 \).

(13 marks)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

\[
I = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx
\]

(Hint: take \( \cos x = \text{Re}(e^{ix}) \), where \( \text{Re} \) indicates the real part, and first calculate \( \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx \) )

(12 marks)

4 (i) The functions \( x(t) \) and \( y(t) \) satisfy the system of differential equations

\[
\begin{cases}
  \dot{x} = x + 2y, \\
  \dot{y} = y + 2x
\end{cases}
\]

(where dot denotes differentiation with respect to \( t \)) and the initial conditions \( x(0) = 4 \) and \( y(0) = 2 \). Use the Laplace transform to find \( x(t) \) and \( y(t) \) for \( t > 0 \).

(15 marks)

(ii) Two functions, \( f(t) \) and \( g(t) \), are defined by

\[
f(t) = \begin{cases}
  e^t, & t \leq 0, \\
  e^{-2t}, & t > 0
\end{cases}, \quad g(t) = \begin{cases}
  e^{2(t-1)}, & t \leq 1, \\
  e^{-3(t-1)}, & t > 1
\end{cases}
\]

(a) Using direct integration, calculate the Fourier transforms, \( F(\omega) \) and \( G(\omega) \), of \( f(t) \) and \( g(t) \).

(b) The convolution of the functions \( f(t) \) and \( g(t) \) is the function \( h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) \, d\tau \). Use the convolution theorem to calculate the Fourier transform \( H(\omega) \) of the function \( h(t) \).

(10 marks)

End of Question Paper
### FORMULA SHEET

#### Table of Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>Region of validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant = $c$</td>
<td>$\frac{c}{s}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$e^{\alpha t}$</td>
<td>$\frac{1}{s-\alpha}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$\cos kt$</td>
<td>$\frac{s}{s^2+k^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$\sin kt$</td>
<td>$\frac{k}{s^2+k^2}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$Re(s) &gt; 0$</td>
</tr>
<tr>
<td>$t^n e^{\alpha t}$</td>
<td>$\frac{n!}{(s-\alpha)^{n+1}}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$e^{\alpha t} \sin kt$</td>
<td>$\frac{k}{(s-\alpha)^2+k^2}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$e^{\alpha t} \cos kt$</td>
<td>$\frac{s-\alpha}{(s-\alpha)^2+k^2}$</td>
<td>$Re(s) &gt; \alpha$</td>
</tr>
<tr>
<td>$\delta(t-T)$</td>
<td>$e^{-sT}$</td>
<td>delta function</td>
</tr>
<tr>
<td>$H(t-T)$</td>
<td>$\frac{e^{-sT}}{s}$</td>
<td>step function</td>
</tr>
<tr>
<td>$H(t) - H(t-T)$</td>
<td>$\frac{1}{s}(1-e^{-sT})$</td>
<td>rectangular pulse</td>
</tr>
</tbody>
</table>

**Note:** in this table the parameters $\alpha$ and $k$ are real constants and $H$ is the Heaviside step function.
Some general properties of the Laplace transform

In the following table the notation $L\{f(t)\} = F(s)$ has been used.

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$L{af(t) + bg(t)} = aL{f(t)} + bL{g(t)}$</td>
</tr>
<tr>
<td>Differentiation w.r.t. $t$</td>
<td>$L{\frac{d}{dt}f(t)} = sF(s) - f(0)$</td>
</tr>
<tr>
<td>Differentiation twice with respect to $t$</td>
<td>$L{\frac{d^2}{dt^2}f(t)} = s^2F(s) - sf(0) - f'(0)$</td>
</tr>
<tr>
<td>Integration</td>
<td>If $g(t) = \int_0^t f(u)du$ then $L{g(t)} = \frac{1}{s}F(s)$</td>
</tr>
<tr>
<td>Differentiation w.r.t. $s$</td>
<td>$L{tf(t)} = -\frac{dF}{ds}$</td>
</tr>
<tr>
<td>Shift</td>
<td>$L{e^{-kt}f(t)} = F(k + s)$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$L{f(at)} = \frac{1}{</td>
</tr>
<tr>
<td>Time Delay</td>
<td>$L{f(t - a)H(t - a)} = e^{-as}F(s)$</td>
</tr>
</tbody>
</table>

Convolution

For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Fourier transform

The Fourier transform $F(\omega)$ of a function $f(t)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$  

The time shift property: the Fourier transform of a function $f(t - T) = e^{-i\omega T} F(\omega)$.  

The scaling property: the Fourier transform of a function $f(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$.  

Residues

The general formula for the residue at a pole, $z_0$, of order $m$ is

$$\frac{1}{(m - 1)!} \lim_{z \to z_0} \left( \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] \right).$$