A1 A plane is given by the equation
\[ 4x + 5y + 7z = 21 \]
and a line by the equation \( \mathbf{r} = (1, 2, 3) + \lambda(1, 2, -2) \), where \( \lambda \) is a real parameter.

(i) Show that the line does not intersect the plane. \( \text{(4 marks)} \)

(ii) Therefore, calculate the distance of the line to the plane. \( \text{(4 marks)} \)

(iii) Find the direction of the line of intersection of the two planes \( x + 3y - z = 5 \) and \( 2(x - y) + 4z = 3 \). \( \text{(5 marks)} \)

A2 Let \( f(t) = t^3 - t^2 \) be a scalar function and \( \mathbf{V} = \left( \frac{1}{t}, t^2, t^3 \right) \) and \( \mathbf{W} = (t, \sin(t), 0) \) be vectors. Find \( \frac{d(f \mathbf{V})}{dt} \), \( \frac{d(\mathbf{V} \cdot \mathbf{W})}{dt} \) and \( \frac{d(\mathbf{V} \times \mathbf{W})}{dt} \). \( \text{(8 marks)} \)

A3 Stokes’ theorem may be written:
\[ \oint_C \mathbf{G} \, d\mathbf{r} = \int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} \, dS \]

Indicate whether the following statements about Stokes’ theorem, as expressed here, are true or false

(i) The term \( \nabla \times \mathbf{G} \) is the curl of the vector field \( \mathbf{G} \).

(ii) The surface \( S \) is surrounded by a closed line \( C \).

(iii) \( \mathbf{n} \) is a unit vector parallel with the boundary \( C \).

(iv) \( \int_S dS \) is a surface integral, over the surface \( S \). \( \text{(4 marks)} \)
Section B

B1  (i) Consider the function

\[ f(x, y) = \tan^{-1} \frac{y}{x} \]

Determine all partial derivatives to first and second order and show that

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \]

Hint: \( \tan^{-1} \) (also called arctan) is the inverse function of the \( \tan \)-function and you are given that

\[ \frac{d \tan^{-1} u}{du} = \frac{1}{1 + u^2}. \]

(12 marks)

(ii) A scalar function is given as

\[ \phi(x, y, z) = x^2 - y \sin(x - z). \]

(a) Calculate the gradient of \( \phi(x, y, z) \), i.e. calculate \( \mathbf{V} = \nabla \phi \).

(3 marks)

(b) Using your result, calculate the divergence of \( \mathbf{V} \).

(4 marks)

(c) By explicit calculation, show that \( \nabla \times \mathbf{V} = 0 \).

(6 marks)
B2 (i) A vector field is given by
\[ \mathbf{V} = r^2 \hat{r} + (a + r) \hat{\theta} + bz \hat{z} \]
in cylindrical polar coordinates, where \( a \) and \( b \) are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as
\[ \nabla \cdot \mathbf{V} = \frac{1}{r} \frac{\partial}{\partial r} (r V_1) + \frac{1}{r} \frac{\partial}{\partial \theta} (V_2) + \frac{\partial}{\partial z} (V_3) \]
and
\[ \nabla \times \mathbf{V} = \frac{1}{r} \begin{vmatrix} \hat{r} & r \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_1 & rV_2 & V_3 \end{vmatrix} \]
respectively. Indicate where the field is irrotational. (10 marks)

(ii) Sketch the region of integration represented by the repeated integral
\[ \int \int_R x y^2 \, dx \, dy \]
where \( R \) is the region such that \( x \geq 0 \), \( y \geq 0 \), and \( x^2 + y^2 \leq a^2 \). By transforming to plane polar coordinates, evaluate the integral. (15 marks)

B3 (i) A particle \( P \) with position vector \( \mathbf{r} \) moves in a plane polar \((r, \theta)\) coordinate system. Write down the relationship between the unit vectors \( \mathbf{i}, \mathbf{j} \) in Cartesian coordinates \((x, y)\) and the unit vectors \( \hat{r}, \hat{\theta} \) in plane polar coordinates \((r, \theta)\). Hence, or otherwise, show that the velocity \( \mathbf{v} \) of the particle may be expressed as
\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = r \hat{r} + r \dot{\theta} \hat{\theta} \]
and find the component of the acceleration of the particle in the \( \hat{\theta} \) direction. (15 marks)

(ii) A magnetic field is given, in cylindrical polar coordinates \((r, \theta, z)\), as
\[ \mathbf{H} = H_0 r^2 \hat{\theta}/a^2 \]
with \( r \leq a \), where \( H_0 \) and \( a \) are positive constants. The magnetic field vanishes for \( r > a \). Evaluate
\[ \oint_C \mathbf{H} \cdot d\mathbf{r}, \]
where \( C \) is the circle \( z = 0, \, r = R \), described in the anticlockwise sense for \( R < a \). (10 marks)

End of Question Paper