1. (i) Find and classify the stationary points of the function

\[ F(x, y) = \frac{1}{y} - \frac{1}{x} - 4x + 9y. \]

(12 marks)

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values attained by the function

\[ f(x, y, z) = 3x + 4y + 5z \]

subject to the constraint

\[ x^2 + y^2 + z^2 = 50. \]

(13 marks)
2  
(i) Sketch the region over which the integral $I_1$ is defined, where

$$I_1 = \int \int_R (3y^2 + 10x^4y) \, dx \, dy,$$

and $R$ is the rectangle with vertices $(0, 0), (1, 0), (1, 2)$ and $(0, 2)$.

Evaluate $I_1$.  
(6 marks)

(ii) Sketch the region over which the integral $I_2$ is defined, where

$$I_2 = \int_0^{\sqrt{\pi}} \int_{x^2}^{\pi} x \cos (x^2) \, dy \, dx.$$

Evaluate $I_2$ by changing the order of integration.  
(10 marks)

(iii) Sketch the region over which the integral $I_3$ is defined, where

$$I_3 = \int \int_D \frac{1}{1 + x^2 + y^2} \, dx \, dy,$$

and $D$ is the region consisting of all points $(x, y)$ such that $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 4$.

Evaluate $I_3$.  
(9 marks)

3  
(i) A scalar field $\phi$ is given by

$$\phi(x, y, z) = 2xz^4 - x^2y.$$

Find grad $\phi$ and $\nabla^2 \phi$ at the point $(2, -2, 1)$.  
(8 marks)

(ii) A vector field $\mathbf{A}$ is given by

$$\mathbf{A} = 2x^2i - 3yzj + xz^2k.$$

Find div $\mathbf{A}$ and curl $\mathbf{A}$.  
(7 marks)

By evaluating both sides, verify that

$$\nabla^2 \mathbf{A} = \text{grad div} \mathbf{A} - \text{curl curl} \mathbf{A}.$$  
(10 marks)
Let $F$ be the vector field

$$F = 3yi + 4xj + 2z^2k.$$ 

Let $C$ be the closed curve given by $x^2 + y^2 = 9$ and $z = 0$ and let $S_1$ be the surface of the open hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$. Let $S_2$ be the disc given by $x^2 + y^2 \leq 9$ and $z = 0$ and let $S$ be the surface of the closed hemisphere formed by $S_1$ and $S_2$. Let $V$ be the volume enclosed by $S$.

(i)  Let $I$ be the line integral 

$$I = \oint_C F.dr$$

and let $J$ be the surface integral 

$$J = \iint_{S_1} \text{curl } F.dS.$$ 

Evaluate $I$ and $J$ separately and verify that, in accordance with Stokes’ Theorem, $I = J$. \hspace{1cm} \text{(16 marks)}

(ii) Let $K$ be the volume integral 

$$K = \iiint_V \text{div } F \, dV.$$ 

Evaluate $K$ and use Gauss’ Divergence Theorem to write down the value of the surface integral 

$$L = \iint_S F.dS.$$ \hspace{1cm} \text{(9 marks)}

End of Question Paper
Formula Sheet for MAS243
Trigonometry

\[ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \]
\[ \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \]
\[ \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \]

\[ a \cos \theta + b \sin \theta = R \cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos \alpha = \frac{a}{R}, \sin \alpha = \frac{b}{R} \]

\[ \cos^2 \theta = \frac{1}{2} (\cos 2\theta + 1) \]
\[ \cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos 3\theta) \]
\[ \cos^4 \theta = \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta) \]
\[ \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \]
\[ \sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta) \]
\[ \sin^4 \theta = \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta) \]