1. (a) Find the line of intersection of the planes

\[ 2x - 3y + 5z = 7 \quad \text{and} \quad 5x - 8y + 7z = 10, \]

giving the equation in parametric form. (9 marks)

(b) Find the Cartesian equation of the plane which contains the points

\( (2,5,-1), (3,1,7), (-1,4,-6) \). (8 marks)

(c) Find the Hermite form of the matrix

\[
\begin{pmatrix}
0 & 1 & -3 & 2 & -5 \\
0 & -2 & 6 & -4 & 9 \\
0 & 3 & -9 & 6 & -10 \\
0 & -4 & 17 & -18 & 13
\end{pmatrix},
\]

specifying each row operation using any appropriate notation. (8 marks)

2. (i) Show that \( (3,9,-13), (3,10,18), (2,6,-9) \) is a basis for \( \mathbb{R}^3 \) and that one of

\( (5,-2), (-10,4) \) and \( (3,4), (4,5) \) is a basis for \( \mathbb{R}^2 \), but the other is not. (8 marks)

(ii) Let \( T \) be the linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) given by the formula

\[ T(x, y, z) = (x + 2y - 3z, 2x + 5y - 2z). \]

Find the matrix of \( T \) with respect to the bases given in part (i) above. (8 marks)

(iii) Find bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \) so that the linear transformation \( T \) of part (ii) above is

in Smith form. (9 marks)
3. A real $8 \times 8$ matrix $A$ has minimum polynomial $x^4 - 2x^3 + x^2 - 12x + 20$. The trace of $A$ is 4. The geometric multiplicity of the eigenvalue 2 is 3.

(i) Find all the eigenvalues of $A$ (5 marks)

(ii) Find the algebraic multiplicity of the complex eigenvalues. (5 marks)

(iii) Write down the invariant factors of $xI - A$. (5 marks)

(iv) Write down the rational form of $A$. (5 marks)

(v) Write down the Jordan form of $A$. (5 marks)

4. Consider the second order differential equation $\frac{d^2 x}{dt^2} + 2 \frac{dx}{dt} + 10x = 0$.

(i) By introducing a new variable $y$, write this equation in the form $\frac{dx}{dt} = Ax$ where $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A$ is a matrix. (5 marks)

(ii) Find the eigenvalues and eigenvectors of $A$. (The eigenvectors should be scaled so that the first coordinate in each case is equal to 1.) (8 marks)

(iii) Hence find a matrix $P$ such that $P^{-1}AP = J$, the Jordan form of $A$. (4 marks)

(iv) Hence solve the differential equation, using the formula $e^{\pm j\theta} = \cos \theta \pm j \sin \theta$ to write the general solution in the form

$$x = Be^{kt} \cos pt + Ce^{-kt} \sin pt$$

where the values of $k$ and $p$ are to be found. (8 marks)
5. (i) Give a matrix representation of the quadratic form

\[ Q(x, y, z) = 15x^2 + 29y^2 + 22z^2 + 12xz + 12yz \]

(5 marks)

(ii) Show that \( \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix} \) is an eigenvector of the matrix and find a normalised eigenvector with the same eigenvalue. (5 marks)

(iii) Find a normalised eigenvector associated with the eigenvalue 22. (5 marks)

(iv) By taking the cross product of the eigenvectors in parts (ii) and (iii), or otherwise, find a third eigenvalue and normalised eigenvector of the matrix. (5 marks)

(v) Hence write \( Q(x, y, z) \) as a sum or difference of squares and describe the surface \( Q(x, y, z) = 11 \). (5 marks)

6. Let \( z = \left( 2x^2 + 3y^2 \right) e^{-x^2-y^2} \).

(i) Show that there are 5 critical points of \( z \) in the \((x, y)\)-plane, and find their coordinates. (11 marks)

(ii) Find the values of \( z \) at the critical points. (3 marks)

(iii) Using the Hessian matrix classify each critical point as a local maximum, a local minimum or a saddle point. (11 marks)