SCHOOL OF MATHEMATICS AND STATISTICS  
Spring semester 2009-2010

Applied Differential Equations  
2 hours

*Attempt all FOUR questions.*
1  (i) Derive the first three non-zero terms of the Taylor series solution to the differential equation
\[ \frac{dy}{dx} = x + y^2 - 2, \quad y(0) = 1. \]  

(5 marks)

(ii) Given that values of \( y \) obtained from this solution are \( y(0.1) = 0.895 \) and \( y(0.2) = 0.78 \), advance the solution to \( x = 0.3 \) by applying the Adams-Bashforth-Moulton method, given by
\[
\begin{align*}
y_{n+1} &= y_n + \frac{h}{2} [3f_n - f_{n-1}], \\
y_{n+1} &= y_n + \frac{h}{2} [f_{n+1} + f_n].
\end{align*}
\]
with formula (1) as the predictor and formula (2) as the corrector. Here \( h \) is the step-size, \( y_i \) is an estimate of \( y(x_i) \) and \( f_i = f(x_i, y_i) = x_i + y_i^2 - 2 \). Use \( h = 0.1 \), and work throughout correct to 4 decimal places.  

(7 marks)

(iii) The following results were obtained by using the above Adams-Bashforth-Moulton method to solve the equation in part (i) as far as \( x = 2 \), with two different step-lengths:

<table>
<thead>
<tr>
<th>( h )</th>
<th>( y(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>-0.496185</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.495792</td>
</tr>
</tbody>
</table>

Use these data to estimate the step-length required to ensure that the global discretization error in \( y(2) \) is smaller than \( 5 \times 10^{-5} \). You may assume that the Adams-Bashfort-Moulton method is of order 2.  

(4 marks)

(iv) The predictor formula (1) can be written in the form
\[
\sum_{j=0}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j}.
\]
Determine the values of \( \alpha_j, \beta_j, j = 0, 1, 2 \). Show that the method is consistent. Write down the first characteristic polynomial for the method and show that the method is stable.  

(5 marks)

(v) Apply the method given by formula (2) to the test equation \( y' = \lambda y \), where \( \lambda \) is a constant, to show that
\[
y_{n+1} = \frac{1 + \frac{h}{2}}{1 - \frac{h}{2}} y_n,
\]
where \( h \) = \( h \lambda \). Hence show that the interval of absolute stability for the method is \((-\infty, 0)\).  

(4 marks)
(i) Use Taylor series to derive the relation
\[ y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h^2), \]
where \( x_{n\pm1} = x_n \pm h \). \( (5 \text{ marks}) \)

(ii) Use the above relation and the relation
\[ y'(x_n) = \frac{y(x_{n+1}) - y(x_{n-1})}{2h} + O(h^2) \]
to show that the differential equation
\[ y''(x) + P(x)y'(x) + Q(x)y(x) = R(x) \]
may be approximated at \( x = x_n \) by the following equation
\[ (1 - \frac{1}{2}hP_n)y_{n-1} - (2 - h^2Q_n)y_n + (1 + \frac{1}{2}hP_n)y_{n+1} = h^2R_n, \]
where \( P_n = P(x_n) \), \( Q_n = Q(x_n) \) and \( R_n = R(x_n) \).

Use the above relation with \( h = 0.25 \) to show that the differential equation
\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2}y = 3, \]
with boundary conditions \( y(1) = 2 \), \( y(2) = 4 \) may be approximated by a system of linear algebraic equations \( A\mathbf{y} = \mathbf{b} \), where \( \mathbf{b} = [-1.6125, 0.1875, -4.0981]^T \). Determine the elements of matrix \( A \) correct to 4 decimal places. Do not attempt to solve these equations. \( (14 \text{ marks}) \)

(iii) The following table contains grid-point values of two solutions \( Y_1(x) \) and \( Y_2(x) \) of a linear differential equation \( d^2y/dx^2 = f(x, y, y') \) obtained using the fourth-order Runge-Kutta method. \( Y_1(x) \) was determined using the initial conditions \( y(1) = 1 \), \( y'(1) = 0 \), and \( Y_2(x) \) was obtained using \( y(1) = 1 \), \( y'(1) = 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_1(x) )</td>
<td>1.07444</td>
<td>1.36280</td>
<td>2.02865</td>
<td>3.41281</td>
</tr>
<tr>
<td>( Y_2(x) )</td>
<td>1.36222</td>
<td>2.05472</td>
<td>3.35550</td>
<td>5.85592</td>
</tr>
</tbody>
</table>

Form a linear combination of these two solutions which is the numerical solution to the equation \( d^2y/dx^2 = f(x, y, y') \) with boundary conditions \( y(1) = 1 \), \( y(2) = 2 \). Calculate the values of this solution at all the \( x \)-values in the table. \( (6 \text{ marks}) \)
3 (i) By using the change of independent variables \( \nu = x + ct, \eta = x - ct \), show that the partial differential equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t)
\]  

may be written as

\[
\frac{\partial^2 u}{\partial \nu \partial \eta} = 0.
\]

Hence determine the general solution of equation (3).  

(11 marks)

(ii) Given that the general solution of equation (3) is in the form

\[
u(x, t) = F(x + ct) + G(x - ct),
\]

where \( F \) and \( G \) are arbitrary functions, use the initial conditions

\[
u(x, 0) = f(x), \quad -\infty < x < \infty,
\]

\[
\frac{\partial u(x, t)}{\partial t} \bigg|_{t=0} = g(x), \quad -\infty < x < \infty,
\]

to show that \( F(x) \) and \( G(x) \) satisfy

\[
f(x) = F(x) + G(x),
\]

\[
g(x) = cF'(x) - cG'(x),
\]

and hence show that

\[
u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi.
\]

(14 marks)
4  (i) The function \( u(x, y) \) satisfies Laplace’s equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y)
\]
in the rectangular region \( 0 \leq x \leq a, 0 \leq y \leq b \). Show that solutions of Laplace’s equation of the form \( X(x)Y(y) \) satisfy the relation
\[
\frac{X''}{X} = -\frac{Y''}{Y} = \alpha,
\]
where \( \alpha \) is a constant. Explain why \( \alpha \) is a constant.  

(4 marks)

(ii) Assuming that \( \alpha = s^2 \) (\( s \neq 0 \) and \( s \) is a real number), show that
\[
X(x) = A \cosh sx + B \sinh sx \quad \text{and} \quad Y(y) = C \cos sy + D \sin sy,
\]
where \( A, B, C, \) and \( D \) are arbitrary constants.  

(4 marks)

(iii) Find the corresponding solutions when \( \alpha = 0 \).  

(4 marks)

(iv) Use the results in part (ii) and (iii) to determine the solution \( u(x, y) \) of the Laplace’s equation which satisfies the boundary conditions
\[
\begin{align*}
u(0, y) &= 0, & u(a, y) &= u_0 \cos(3\pi y/b), \quad (0 \leq y \leq b), \\
\left. \frac{\partial u(x, y)}{\partial y} \right|_{y=0} &= 0, & \left. \frac{\partial u(x, y)}{\partial y} \right|_{y=b} &= 0, \quad (0 \leq x \leq a),
\end{align*}
\]
where \( u_0 \) is a constant.  

(13 marks)

You may assume that only trivial solutions result from the choice \( \alpha < 0 \).

End of Question Paper