The positive random variables $X_1$, $X_2$, $\ldots$, $X_n$ are independent observations from the following density function

$$\frac{x^2}{2\eta^3}e^{-\frac{x}{\eta}} \quad (x > 0)$$

which depends on the unknown parameter $\eta \in (0, \infty)$.

(a) Give the relationship between this distribution and one of the standard ones used in the course. Give the mean and variance of the distribution in terms of $\eta$.  

(b) Show that $X/3$ is the maximum likelihood estimator of $\eta$, where $X$ is the average of the $X$'s. 

(c) Show that the $k$-unit likelihood region is the set of $\eta$ for which

$$\left(\frac{\hat{\eta}}{\eta} - 1\right) - \log\left(\frac{\hat{\eta}}{\eta}\right) \leq \frac{k}{3n},$$

where $\hat{\eta}$ is the MLE of $\eta$. 

(d) Show that the maximum likelihood estimator of $\eta$ is unbiased and that its variance is $\eta^2/(3n)$.

(e) Give the maximum likelihood estimator of $\eta^2$. Find the bias of this estimator. Propose an unbiased estimator of $\eta^2$. 

1
The observations $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed and

$$Y_i \sim Ge \left( \frac{e^{-\gamma}}{1 + e^{-\gamma}} \right),$$

where $\gamma \in (-\infty, \infty)$ is an unknown parameter. [Thus $Y_i$ counts the number of failures before the first success in Bernoulli trials.]

(a) (i) Show that $E\overline{Y} = e^\gamma$, where $\overline{Y}$ is the average of the $Y$’s. \hspace{1cm} (3 marks)

(ii) Show that the log-likelihood is

$$n \left[ -(\overline{y} + 1) \log(1 + e^{-\gamma}) - \gamma \right],$$

justifying your manipulations carefully. \hspace{1cm} (6 marks)

(iii) Show that the (Fisher) information concerning $\gamma$ is

$$\frac{n}{(1 + e^{-\gamma})},$$

Hence, give the asymptotic distribution of the MLE of $\gamma$. \hspace{1cm} (15 marks)

(b) (i) Use the Neyman-Pearson Lemma to show that the form of the best test of the null hypothesis that $\gamma = 0$ against the alternative that $\gamma = \gamma_1 > 0$ (for some specific $\gamma_1$) is ‘Reject $\gamma = 0$ if $\overline{Y} \geq k$’ — again you should justify your manipulations carefully. \hspace{1cm} (7 marks)

(ii) Suppose that the null hypothesis $\gamma = 0$ is to be tested against $\gamma > 0$. Show that there is a uniformly most powerful test. \hspace{1cm} (2 marks)
Suppose that $Y \sim N(\Theta, 5)$ and $Y = \log X$. Suppose also that $Y_1 = \log X_1$, $Y_2 = \log X_2$, \ldots, $Y_n = \log X_n$ are independent identically distributed random variables each with the same distribution as $Y$. Suppose that the prior beliefs about $\Theta$ are that $\Theta$ is $N(2, 1)$.

(a) Give $EX$ in terms of $\Theta$. The scientist is concerned with assessing the evidence for $EX > 99.5$. Express this as a statement about $\Theta$. 

(b) If the values of the $Y$’s are $y_1, y_2, \ldots, y_n$, give the posterior distribution for $\Theta$.

(c) Suppose $n = 3$, $y_1 = 4.1$, $y_2 = 7.7$ and $y_3 = 6.2$. Describe how these data change the degree of belief about the hypothesis that $EX > 99.5$.

(d) Using the data just given, the scientist wants to offer a point inference about $EX$. He says that the loss in making the inference $b$ is $(b - EX)^2$. What point estimate should be used? Justify the form of the estimate you propose.
The random variables \( X_i, i = 1, 2, \ldots, n, \) are independent and

\[ X_i \sim N \left( 0, \frac{1}{2\nu a_i^2} \right), \]

where the \( a_i \) are known constants and \( \nu \) is an unknown parameter. Let

\[ T(x) = \sum_{i=1}^{n} (a_i x_i)^2. \]

(a) Show that

\[ \nu^{n/2} \exp(-\nu T(x)) \]

is a version of the likelihood, justifying your manipulations carefully. Explain why it is described as ‘a version of the likelihood’.  

(b) Obtain the MLE of \( \nu \).

(c) Suppose the prior on \( \nu \) is \( \text{Ga}(b+1, c) \), with \( b > 0 \) and \( c > 0 \) known constants.

(i) Show that the mode of the prior is \( b/c \).

(ii) Find the posterior for \( \nu \) and give the posterior mode (in terms of \( T(x) \), \( b \) and \( c \)).

(iii) Indicate with a sketch what is meant in this context by the highest density \( \alpha \% \) credible region for the posterior. What value is in this interval for every \( \alpha > 0 \)?

(d) Show that the posterior mode can be written as

\[ w \times \text{prior mode} + (1 - w) \times \text{MLE} \]

for \( w = c/(c + T(x)) \).

End of Question Paper