Data Provided: Formula Sheet

The University Of Sheffield.

MAS242

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2010-2011

Mathematics III(Electrical)

2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 z is the complex number $x + jy$.

(i) Sketch the four regions in the z-plane corresponding to $x \geq 1$, $y \geq x + 4$, $|z| \leq 3$ and $|z + 2| \leq 1$. ***(8 marks)***

(ii) For the mapping $w = (4 + 3j)z + j$ where $w = u + jv$,

(a) find $u(x, y)$ and $v(x, y)$;

(b) find the image in the w-plane of the region $|z| \leq 1$ in the z-plane;

(c) sketch your results in the z- and w-planes. ***(13 marks)***

(iii) Find out if the image of the circle $|z| = \sqrt{2}$ under the bilinear mapping $w = \dfrac{4z + 1 - 3j}{z - 1 - j}$ is a circle or a straight line. ***(4 marks)***

2 Expand the function $f(z) = \dfrac{z}{(z + j)(z - 2j)}$ into partial fractions and hence

(i) find the first three non-zero terms of the power series expansion of $f(z)$ about the point $z = 0$ using either the Taylor series or the binomial expansion method, Show the region of convergence of the power series and all poles and zeros of $f(z)$ on the Argand diagram. ***(16 marks)***

(ii) Find the first four terms of the Laurent series expansion of $f(z)$ about the point $z = -j$. ***(9 marks)***

3 (i) Find all the poles and zeros of $f(z) = \dfrac{z}{(9 - z^2)(z + j)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, where

(a) C is the circle $|z| = \dfrac{1}{2}$

(b) C is the circle $|z + 3| = 5$.

(16 marks)

(ii) *(Note that the question on the actual paper contained a typo - this is corrected here.)* By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2 - 4x + 5} dx.$$

(9 marks)

4 (i) The function $x(t)$ satisfies the differential equation

$$\ddot{x} - 4\dot{x} + 3x = te^{-2t}$$

(where dot denotes differentiation with respect to t) and the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Show that the Laplace transform, $X(s)$, of $x(t)$ is given by

$$X(s) = \frac{1}{50(s-3)} - \frac{1}{18(s-1)} + \frac{1}{15(s+2)^2} + \frac{8}{225(s+2)}$$

Hence determine $x(t)$ for $t > 0$. ***(13 marks)***

(ii) Sketch the functions $f(t) = H(t)e^{-\alpha t}$ and $g(t) = H(t)e^{-\beta t}$ for $\beta > \alpha > 0$ where $H(t)$ is the step function. Using direct integration, show that the Fourier transform of $f(t)$ is $\dfrac{1}{\alpha + j\omega}$. Write $g(t)$ in terms of $f(t)$, and hence verify the scaling property of the Fourier Transform for this function.

(12 marks)

End of Question Paper