1. \( z \) is the complex number \( x + jy \).
   
   (i) Sketch the four regions in the \( z \)-plane corresponding to \( x \geq 1, y \geq x + 4, |z| \leq 3 \) and \( |z + 2| \leq 1 \). (8 marks)
   
   (ii) For the mapping \( w = (4 + 3j)z + j \) where \( w = u + jv \),
        
        (a) find \( u(x, y) \) and \( v(x, y) \);
        
        (b) find the image in the \( w \)-plane of the region \( |z| \leq 1 \) in the \( z \)-plane;
        
        (c) sketch your results in the \( z \)- and \( w \)-planes. (13 marks)
   
   (iii) Find out if the image of the circle \( |z| = \sqrt{2} \) under the bilinear mapping
        
        \[ w = \frac{4z + 1 - 3j}{z - 1 - j} \] is a circle or a straight line. (4 marks)

2. Expand the function \( f(z) = \frac{z}{(z + j)(z - 2j)} \) into partial fractions and hence
   
   (i) find the first three non-zero terms of the power series expansion of \( f(z) \)
       about the point \( z = 0 \) using either the Taylor series or the binomial expansion method, Show the region of convergence of the power series and all poles and zeros of \( f(z) \) on the Argand diagram. (16 marks)
   
   (ii) Find the first four terms of the Laurent series expansion of \( f(z) \) about the point \( z = -j \). (9 marks)
3 (i) Find all the poles and zeros of \( f(z) = \frac{z}{(9 - z^2)(z + j)} \) and plot them on an Argand diagram. Hence evaluate the integral \( \oint_C f(z) \, dz \), writing your solutions in the form \( a + jb \), where \( a \) and \( b \) are real, where
(a) \( C \) is the circle \( |z| = \frac{1}{2} \)
(b) \( C \) is the circle \( |z + 3| = 5 \).

(16 marks)

(ii) (Note that the question on the actual paper contained a typo - this is corrected here.) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral
\[
I = \int_{-\infty}^{\infty} \frac{1}{x^2 - 4x + 5} \, dx.
\]

(9 marks)

4 (i) The function \( x(t) \) satisfies the differential equation
\[
\ddot{x} - 4\dot{x} + 3x = te^{-2t}
\]
(where dot denotes differentiation with respect to \( t \)) and the initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = 0 \). Show that the Laplace transform, \( X(s) \), of \( x(t) \) is given by
\[
X(s) = \frac{1}{50(s - 3)} - \frac{1}{18(s - 1)} + \frac{1}{15(s + 2)^2} + \frac{8}{225(s + 2)}
\]
Hence determine \( x(t) \) for \( t > 0 \).

(13 marks)

(ii) Sketch the functions \( f(t) = H(t)e^{-\alpha t} \) and \( g(t) = H(t)e^{-\beta t} \) for \( \beta > \alpha > 0 \) where \( H(t) \) is the step function. Using direct integration, show that the Fourier transform of \( f(t) \) is \( \frac{1}{\alpha + j\omega} \). Write \( g(t) \) in terms of \( f(t) \), and hence verify the scaling property of the Fourier Transform for this function.

(12 marks)

End of Question Paper