The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010-2011**

Mathematics III (Control) **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1. (i) Find the general solution of the equations

$$x_1 + 2x_3 + 3x_4 = 7$$
$$2x_1 + 3x_2 - 2x_3 + 19x_4 = 3$$
$$3x_1 - 2x_2 + 10x_3 = 29$$

giving the equation in parametric form. ***(11 marks)***

(ii) Find the Cartesian equation of the plane whose parametric equation is

$$(x, y, z) = (3,1,-2) + s(2,1,4) + t(1,-3,5).$$

(6 marks)

(iii) Show that the planes

$$3x + 4y + z = 8$$
$$2x + 3y - z = 4$$
$$x + 7z = a$$

have no point in common unless a has a specific value, which you should find. How do the planes intersect when a has this value? ***(8 marks)***

2. Let $B = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 4 & 5 & 0 & 1 & 0 \\ -2 & -3 & -5 & -5 & -1 & 0 & 0 & 1 \end{pmatrix}$

(i) Reduce the matrix B to Hermite form. ***(6 marks)***

(ii) Hence explain why $(1,3,-2), (0,1,-3), (0,2,-5)$ is a basis for $\mathbb{R}^3$ but $(1,3,-2), (0,1,-3), (1,4,-5)$ is not.
Write $(2,5,-1)$ as a linear combination of the basis vectors above. ***(4 marks)***

(iii) If T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that

$$T(1,3,-2) = (-9,-2)$$
$$T(0,1,-3) = (0,2)$$
$$T(0,2,-5) = (-1,3)$$

then what is $T(2,5,-1)$? ***(4 marks)***

(iv) Find the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$. ***(6 marks)***

(v) Find a non-zero vector in the kernel of T. ***(5 marks)***

3. Let the linear operator $T: \mathbb{R}^4 \to \mathbb{R}^4$ be represented by the matrix

$$A = \begin{pmatrix} -1 & -2 & -2 & -2 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & -3 & -1 \end{pmatrix}.$$

(i) Show that $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A and find the associated eigenvalue.

(4 marks)

(ii) Find a non-zero fixed point of T and express this result in terms of an eigenvalue and an eigenvector. ***(8 marks)***

(iii) Write down the trace and determinant of A. ***(2 marks)***

(iv) Given that $1+\mathrm{j}$ is an eigenvalue of A (you need **not** prove this), find the fourth eigenvalue of A. ***(2 marks)***

(v) Write down the characteristic polynomial of A and use the Cayley-Hamilton theorem to evaluate the matrix $A^4 - 3A^3 + 4A^2 - A + I$. ***(5 marks)***

(vi) Find the rational form of A. ***(4 marks)***

4. Let A be the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 7 & 2 \end{pmatrix}.$$

(i) Find the eigenvalues of A and state the algebraic multiplicity of each.
(*Hint*: the eigenvalues are integers between -5 and 5.)

(5 marks)

(ii) Find the eigenspace associated with each eigenvalue and hence state the geometric multiplicity of each eigenvalue.

(6 marks)

(iii) Write down the Jordan form J of A in such a way that the starred entries in $J = \begin{pmatrix} * & \cdot & \cdot \\ \cdot & * & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ are not equal.

(2 marks)

(iv) Show that $J = P^{-1}AP$, where

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 4 & -1 & 1 \\ 16 & 1 & -2 \end{pmatrix}.$$

(4 marks)

(v) Hence or otherwise find the general solution of the differential equation

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = 2\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 7\frac{\mathrm{d}x}{\mathrm{d}t} + 4x.$$

(8 marks)

5. (i) Give a matrix representation of the quadratic form

$$Q(x, y, z) = 11x^2 - 2y^2 + 18z^2 - 24xy + 16xz + 32yz$$

(5 marks)

(ii) Show that $\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$ is an eigenvector of the matrix and find a normalised eigenvector with the same eigenvalue. ***(5 marks)***

(iii) Find a normalised eigenvector associated with the eigenvalue 18. ***(5 marks)***

(iv) By taking the cross product of the eigenvectors in parts (ii) and (iii), or otherwise, find a third eigenvalue and normalised eigenvector of the matrix. ***(5 marks)***

(v) Hence write $Q(x, y, z)$ as a sum or difference of squares and describe the surface $Q(x, y, z) = 18$. ***(5 marks)***

6. (i) Let $z = f(x, y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

(a) Write down the two equations which must be satisfied at any critical point of $f(x, y)$. By adding these show that at a critical point either $x + y = 0$ or $xy = 9$. ***(5 marks)***

(b) Find all the critical points of $f(x, y)$ and the values of z at these points. ***(7 marks)***

(c) For each critical point use an appropriate second derivative test to see whether it is a local maximum, a local minimum or a saddle point. ***(7 marks)***

(ii) Use the method of Lagrange multipliers to find the minimum value of the function $f(x, y) = 2x^2 - 2xy + y^2$ on the line $x + y = 20$. ***(6 marks)***

End of Question Paper