Data provided: Graph paper, Laplace Transform table

MAS312

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

Classical Control Theory **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 The transfer functions of two linear systems are

$$G_1(s) = \frac{V(s)}{U(s)} = \frac{s+1}{s^2+2s+5} \quad \text{and} \quad G_2(s) = \frac{Y(s)}{V(s)} = \frac{s}{s^2+5s+6} \, .$$

(i) Give the two linear differential equations relating the variables $u(t)$, $v(t)$; and $v(t)$, $y(t)$. Give the impulse responses of the systems G_1 and G_2.

(8 marks)

(ii) For the series connection $G = G_1G_2$ compute the overall transfer function $\dfrac{Y(s)}{U(s)}$. Determine the open-loop poles and zeros, and establish whether the system is open-loop stable.

(3 marks)

(iii) Now close the loop to obtain a constant-gain negative feedback system. Calculate the closed-loop transfer function $H(s)$ and the characteristic equation for the closed-loop poles.

Use the Routh-Hurwitz criterion to show that the closed-loop system is stable for all gain values $k \geq 0$. ***(7 marks)***

(iv) Hence, sketch the root locus plot; computing the relevant angles of departure and using the mid-point approximation for a breakaway point.

(7 marks)

2 (i) Consider the unity negative feedback system with open-loop transfer function

$$G(s) = \frac{1}{s(s+2)}.$$

Compute the closed-loop transfer function $H(s)$ and the error transfer function, $E(s)/R(s)$, where $r(t)$ is the reference input and $e(t) = r(t) - y(t)$ is the error.

(5 marks)

(ii) Using the Partial Fractions Expansion, compute the error $e(t)$, $t \geq 0$, for the cases where the reference input is:

(a) a unit step, $r(t) = h(t)$ and (b) a unit ramp $r(t) = t\, h(t)$

where $h(t)$ is the Heaviside function.

Show that the steady-state error, $e_{ss} = \lim_{t \to \infty} e(t)$, is zero for the unit step, but is equal to a constant for a unit ramp. Calculate this constant.

(11 marks)

(iii) Now explore the claim that the modified open-loop transfer

$$\tilde{G}(s) = \frac{s+1}{s^2(s+2)}$$

can track both steps and ramps with *zero steady-state error.*

First check, using the Routh–Hurwitz criterion, that the closed-loop transfer function $E(s)/R(s)$ is stable.

Next, appealing to the *form* of the partial fractions expansion (*without computing all the coefficients*), show that the steady-state error to both a unit step and a unit ramp is zero.

Finally, for $h(t)$ the Heaviside function, show that the steady-state error to the *parabolic* input

$$r(t) = \frac{t^2}{2}\, h(t)$$

is a constant. Calculate this constant.

(9 marks)

3 (i) Draw the root locus plot for the constant-gain negative feedback system with open-loop transfer function

$$G(s) = \frac{s-1}{s(s+2)((s+1)^2+4)}$$

and hence conclude that the closed-loop system is unstable for all values of the gain $k > 0$.

(11 marks)

(ii) We aim to modify the above system so that it is stable for *some* values of k. Let us introduce a *second unstable* open-loop zero:

$$\tilde{G}(s) = \frac{(s-1)(s-0.5)}{s(s+2)((s+1)^2+4)}.$$

Using the Routh–Hurwitz table, decide whether this strategy is successful and, if it is, give the range of gain values for which we get closed-loop stability.

(11 marks)

(iii) Give a quick argument (using the root locus plot, for example), to show that the system with an extra *stable open-loop pole*,

$$\bar{G}(s) = \frac{s-1}{s(s+1)(s+2)((s+1)^2+4)},$$

will not work (i.e., the closed-loop system will be unstable for all $k \geq 0$.)

(3 marks)

4 (i) You are given the linear differential equation

$$2\ddot{y} + 8\dot{y} + 6y = 3\dot{u} + u \ .$$

Solve for the output $y(t)$, $t \geq 0$, when $y(0) = 0$, $\dot{y}(0) = 1$ and the input

$$u(t) = \cos(2t)\ h(t) \ .$$

(10 marks)

(ii) Sketch the Nyquist plot for the constant-gain negative feedback system, with open-loop transfer function

$$G(s) = \frac{(s+8)}{(s^3 + 2s^2 + 5s + 8)} \ .$$

Check that the closed-loop system with gain $k_0 = 0.25$ is stable and find the gain and phase margins for this nominal gain value k_0. [The equation you will find when computing the phase margin frequency is cubic in ω^2 and should be:

$$16\omega^6 - 96\omega^4 - 113\omega^2 + 960 = 0 \ ;$$

you do not need to solve this equation, the real root you need is $\omega^2 \simeq 5.0377$.]

(12 marks)

(iii) Suppose now that the actual transfer function has some unmodelled high–frequency dynamics, so that the transfer function is

$$\tilde{G}(s) = \frac{10(s+8)}{(s+10)(s^3 + 2s^2 + 5s + 8)} \ .$$

Decide whether the additional phase contribution makes the system unstable.

(3 marks)

5 (i) Sketch the Nyquist plot of the transfer function

$$G(s) = \frac{s+10}{(s+1)(s+2)(s+3)},$$

determining all frequencies at crossings of the *real* axis. ***(9 marks)***

(ii) This linear system $G(s)$ is placed in a constant-gain negative feedback configuration.

Sketch the root locus plot, specifying all the crossover points on the imaginary axis. These should be computed from the Routh–Hurwitz table. You can use the mid-point approximation for any breakaway points.

Confirm that the critical value of the gain, k_c, and the critical frequencies, $\pm\omega_c$, at crossover, are the same as those obtained from the Nyquist plot and the application of the Nyquist stability criterion.

(12 marks)

(iii) If, instead of the given $G(s)$, we had

$$\tilde{G}(s) = \frac{s+d}{(s+1)(s+2)(s+3)},$$

where d is some parameter, use the centre of asymptotes formula to argue that the closed-loop system is stable for all gain values $k \geq 0$, provided $d < 6$.

(4 marks)

Table of Laplace Transform Pairs

Time Function	Laplace Transform
$h(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{t^n e^{-at}}{n!}$	$\frac{1}{(s+a)^{n+1}}$
$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

End of Question Paper