SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2010-11

Differential Equations: Case Studies in Applied Mathematics

Two hours

Marks will be awarded for your best FOUR answers

A list of basic formulae and theorems for use as necessary is provided on the final sheet of the exam paper.

1. Consider a system of equations

\[ \dot{x} = y + xF(r), \quad \dot{y} = -x + yF(r). \] (*)

where \( r^2 = x^2 + y^2. \)

(i) Use the variable substitution

\[ x = r \cos \theta, \quad y = r \sin \theta, \]

to obtain the system of equations for \( r \) and \( \theta, \)

\[ \dot{r} = rF(r), \quad \dot{\theta} = -1. \]

Thus show that system (*) has a periodic solution for each value of \( r_0 \) such that \( F(r_0) = 0. \) (10 marks)

(ii) By considering a small perturbation, \( r = r_0 + \delta, \) about \( r_0 \) and using a Taylor expansion of function \( F(r), \) show that this periodic solution is a stable limit cycle in the case \( F'(r_0) < 0, \) and it is an unstable limit cycle in the case \( F'(r_0) > 0. \) (8 marks)

(iii) For the case

\[ F(r) = -(r - 1)(r^2 - 7r + 12) \]

find all the limit cycles and determine their stability. (7 marks)

Turn Over
A two-species-in-symbiosis system is described by the system of equations

\[
\frac{dx}{dt} = N_0 x \left(1 - \frac{x}{K_0} + \frac{y}{K_1}\right),
\]

\[
\frac{dy}{dt} = N_1 y \left(1 + \frac{x}{K_2} - \frac{y}{K_3}\right),
\]

where all the parameters are positive.

(i) Use the variable substitution

\[ X = \frac{x}{K_0}, \quad Y = \frac{y}{K_3}, \quad T = N_0 t, \]

to rewrite this system in a dimensionless form

\[
\frac{dX}{dT} = X(1 - X + \beta_0 Y) \equiv f(X,Y),
\]

\[
\frac{dY}{dT} = \rho Y(1 - Y + \beta_1 X) \equiv g(X,Y),
\]

where \( \rho = N_1/N_0, \beta_0 = K_3/K_1 \) and \( \beta_1 = K_0/K_2 \). \( 4 \text{ marks} \)

(ii) Find all critical points of this system. Which critical point corresponds to the successful symbiosis (that is, a non-zero state for both species)? What is the condition necessary for this critical point to be physically possible? \( 9 \text{ marks} \)

(iii) Classify the critical point corresponding to the successful symbiosis. State if it is stable or unstable. \( 12 \text{ marks} \)
A metal bar of length $a$ has the temperature at one end held at 0°C for all $t > 0$, while its other end is thermally insulated. Hence the temperature in the bar, $\Theta$, satisfies the boundary conditions

$$\Theta(0, t) = 0, \quad \frac{\partial \Theta}{\partial x}(a, t) = 0 \quad \text{for all } t > 0. \quad (\ast)$$

The temperature distribution along the bar at $t = 0$ is given by

$$\Theta(x, 0) = \Theta_0 \left( \sin \frac{\pi x}{2a} + 0.01 \sin \frac{5\pi x}{2a} \right). \quad (\dagger)$$

(i) Use the separation variable techniques to find the general solution of the diffusion equation

$$\frac{\partial \Theta}{\partial t} = D \frac{\partial^2 \Theta}{\partial x^2}$$

satisfying the boundary conditions $(\ast)$. \hspace{1cm} (20\,\text{marks})

(ii) Use the results of (i) to obtain the solution satisfying the initial condition $(\dagger)$. \hspace{1cm} (2\,\text{marks})

(iii) You are given that $D = 2 \times 10^{-4}$ m$^2$/s, $a = 1$ m, and $\Theta_0 = 50$° C. Use the fact that the first term in $(\dagger)$ is much larger than the second term to estimate how long it takes for the maximum temperature along the bar to decay to 1° C. \hspace{1cm} (3\,\text{marks})
A reaction-diffusion equation describing locally the calcium-stimulated-calcium-release mechanism is
\[
\frac{\partial u}{\partial t} = -A(u - u_1)(u - u_2)(u - u_3) + D \frac{\partial^2 u}{\partial x^2},
\]
where \( D > 0 \) is the diffusion coefficient, \( A, u_1, u_2 \) and \( u_3 \) are positive constants, and \( 0 < u_1 < u_2 < u_3 \).

(i) Using our standard technique of assuming a wave solution \( u(x, t) = U(z) \) with \( z = x + ct, \ c > 0 \), show that \( U(z) \) satisfies the second-order ordinary differential equation for the wave profile \( U(z) \),
\[
DU'' = cU' + A(U - u_1)(U - u_2)(U - u_3). \tag{*}
\]
Introducing \( V = U' \) rewrite this equation as the system of two first-order differential equations. Find the critical points of this system. \((5 \text{ marks})\)

(ii) You are given that any solution of the first-order equation
\[
U' = -\alpha(U - u_1)(U - u_3), \tag{\dagger}
\]
where \( \alpha \) is a constant, satisfies equation \((*)\). Use this condition to determine \( \alpha \) and \( c \). \((10 \text{ marks})\)

(iii) Find the solution of equation \((\dagger)\) (which is also a solution of equation \((*)\)) that satisfies the boundary conditions \( U \to u_2 \) as \( z \to -\infty \) and \( U \to u_3 \) as \( z \to \infty \). \((10 \text{ marks})\)
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(i) Prove that, if $f$ is independent of $x$, i.e. $f = f(y, y')$, then

$$f - y' \frac{\partial f}{\partial y'} = \text{const}$$

is a first integral of the Euler-Lagrange equation. \hfill (5 marks)

(ii) The Fermat principle states that light propagates between two points, $A$ and $B$, along a path that minimizes the travel time. The speed of light in a medium is $c/n$, where $c = \text{const}$ is the speed of light in empty space, and $n \geq 1$ is the refraction index of the medium. Hence, the ray of the light is an extremal of the functional

$$I = \int_A^B n \ ds,$$

where $s$ is the length along the ray. In particular, when $A$ and $B$ are in the $xy$-plane, their coordinates are $A(x_0, y_0)$ and $B(x_1, y_1)$, $x_0 < x_1$, and $n = n(x, y)$, then

$$I = \int_{x_0}^{x_1} n(x, y) \sqrt{1 + y'^2} \ dx.$$

(a) You are given that $n$ is independent of $x$. Show that the ray of the light is given by $y = y(x)$, where $y(x)$ is a solution of equation

$$n = C \sqrt{1 + y'^2},$$

where $C$ is a constant. \hfill (5 marks)

(b) You are given that $n = \sqrt{1 + (y/a)^2}$, where $a > 0$ is a constant. Find the equation of the family of all rays that pass through the coordinate origin. Consider the cases $C = 1$ and $C \neq 1$ separately.

(You can take without proof that $\int dy/\sqrt{y^2 + h^2} = \sinh^{-1}(y/h)$, where $\sinh^{-1}$ is the function inverse to sinh and $h$ is a constant.) \hfill (15 marks)

End of Question Paper
List of Basic Formulae and Theorems

**Theorem 1:** If a periodic solution of the system of equations

\[ \dot{x} = f(x, y), \quad \dot{y} = g(x, y) \]

exists in a simply connected region, then \( f_x + g_y = 0 \) somewhere in that region.

**Corollary:** There are no periodic solutions in any simply connected region where \( f_x + g_y \neq 0 \) everywhere.

**Theorem 2:** The orbit \( C \) of a periodic solution must enclose at least one critical point.

**Orthogonality conditions for trig functions**

\[ \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when} \quad m \neq n. \]

\[ \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0. \]

**Extremals of functional**

\[ J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx \]

are the solutions to the Euler-Lagrange equation

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0. \]