The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-11**

Differential Equations: Case Studies in Applied Mathematics **Two hours**

Marks will be awarded for your best FOUR answers

A list of basic formulae and theorems for use as necessary is provided on the final sheet of the exam paper.

1 Consider a system of equations

$$\dot{x} = y + xF(r), \qquad \dot{y} = -x + yF(r). \qquad (*)$$

where $r^2 = x^2 + y^2$.

(i) Use the variable substitution

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

to obtain the system of equations for r and θ,

$$\dot{r} = rF(r), \qquad \dot{\theta} = -1.$$

Thus show that system $(*)$ has a periodic solution for each value of r_0 such that $F(r_0) = 0$. ***(10 marks)***

(ii) By considering a small perturbation, $r = r_0 + \delta$, about r_0 and using a Taylor expansion of function $F(r)$, show that this periodic solution is a stable limit cycle in the case $F'(r_0) < 0$, and it is an unstable limit cycle in the case $F'(r_0) > 0$. ***(8 marks)***

(iii) For the case

$$F(r) = -(r-1)(r^2 - 7r + 12)$$

find all the limit cycles and determine their stability. ***(7 marks)***

2 A two-species-in-symbiosis system is described by the system of equations

$$\begin{aligned} \frac{dx}{dt} &= N_0\, x\left(1 - \frac{x}{K_0} + \frac{y}{K_1}\right), \\ \frac{dy}{dt} &= N_1\, y\left(1 + \frac{x}{K_2} - \frac{y}{K_3}\right), \end{aligned}$$

where all the parameters are positive.

(i) Use the variable substitution

$$X = \frac{x}{K_0}, \quad Y = \frac{y}{K_3}, \quad T = N_0 t,$$

to rewrite this system in a dimensionless form

$$\begin{aligned} \frac{dX}{dT} &= X(1 - X + \beta_0 Y) \equiv f(X, Y), \\ \frac{dY}{dT} &= \rho Y(1 - Y + \beta_1 X) \equiv g(X, Y), \end{aligned}$$

where $\rho = N_1/N_0$, $\beta_0 = K_3/K_1$ and $\beta_1 = K_0/K_2$. ***(4 marks)***

(ii) Find all critical points of this system. Which critical point corresponds to the successful symbiosis (that is, a non-zero state for both species)? What is the condition necessary for this critical point to be physically possible? ***(9 marks)***

(iii) Classify the critical point corresponding to the successful symbiosis. State if it is stable or unstable. ***(12 marks)***

3 A metal bar of length a has the temperature at one end held at 0° C for all $t > 0$, while its other end is thermally insulated. Hence the temperature in the bar, Θ, satisfies the boundary conditions

$$\Theta(0,t) = 0, \quad \frac{\partial \Theta}{\partial x}(a,t) = 0 \quad \text{for all} \quad t > 0. \tag{$*$}$$

The temperature distribution along the bar at $t = 0$ is given by

$$\Theta(x,0) = \Theta_0 \left(\sin \frac{\pi x}{2a} + 0.01 \sin \frac{5\pi x}{2a} \right). \tag{$\dagger$}$$

(i) Use the separation variable techniques to find the general solution of the diffusion equation

$$\frac{\partial \Theta}{\partial t} = D \frac{\partial^2 \Theta}{\partial x^2}$$

satisfying the boundary conditions ($*$). ***(20 marks)***

(ii) Use the results of (i) to obtain the solution satisfying the initial condition ($\dagger$). ***(2 marks)***

(iii) You are given that $D = 2 \times 10^{-4}$ m^2/s, $a = 1$ m, and $\Theta_0 = 50^\circ$ C. Use the fact that the first term in ($\dagger$) is much larger than the second term to estimate how long it takes for the maximum temperature along the bar to decay to 1° C. ***(3 marks)***

4 A reaction-diffusion equation describing locally the calcium-stimulated-calcium-release mechanism is

$$\frac{\partial u}{\partial t} = -A(u-u_1)(u-u_2)(u-u_3) + D\frac{\partial^2 u}{\partial x^2},$$

where $D > 0$ is the diffusion coefficient, A, u_1, u_2 and u_3 are positive constants, and $0 < u_1 < u_2 < u_3$.

(i) Using our standard technique of assuming a wave solution $u(x,t) = U(z)$ with $z = x + ct$, $c > 0$, show that $U(z)$ satisfies the second-order ordinary differential equation for the wave profile $U(z)$,

$$DU'' = cU' + A(U-u_1)(U-u_2)(U-u_3). \qquad (*)$$

Introducing $V = U'$ rewrite this equation as the system of two first-order differential equations. Find the critical points of this system. ***(5 marks)***

(ii) You are given that any solution of the first-order equation

$$U' = -\alpha(U-u_1)(U-u_3), \qquad (\dagger)$$

where α is a constant, satisfies equation $(*)$. Use this condition to determine α and c. ***(10 marks)***

(iii) Find the solution of equation $(\dagger)$ (which is also a solution of equation $(*)$) that satisfies the boundary conditions $U \to u_1$ as $z \to -\infty$ and $U \to u_3$ as $z \to \infty$. ***(10 marks)***

5 (i) Prove that, if f is independent of x, i.e. $f = f(y, y')$, then

$$f - y'\frac{\partial f}{\partial y'} = \text{const}$$

is a first integral of the Euler-Lagrange equation. ***(5 marks)***

(ii) The Fermat principle states that light propagates between two points, A and B, along a path that minimizes the travel time. The speed of light in a medium is c/n, where $c = \text{const}$ is the speed of light in empty space, and $n \geq 1$ is the refraction index of the medium. Hence, the ray of the light is an extremal of the functional

$$I = \int_A^B n\, ds, \qquad (*)$$

where s is the length along the ray. In particular, when A and B are in the xy-plane, their coordinates are $A(x_0, y_0)$ and $B(x_1, y_1)$, $x_0 < x_1$, and $n = n(x, y)$, then

$$I = \int_{x_0}^{x_1} n(x, y)\sqrt{1 + y'^2}\, dx.$$

(a) You are given that n is independent of x. Show that the ray of the light is given by $y = y(x)$, where $y(x)$ is a solution of equation

$$n = C\sqrt{1 + y'^2}, \qquad (\dagger)$$

where C is a constant. ***(5 marks)***

(b) You are given that $n = \sqrt{1 + (y/a)^2}$, where $a > 0$ is a constant. Find the equation of the family of all rays that pass through the coordinate origin. Consider the cases $C = 1$ and $C \neq 1$ separately. (You can take without proof that $\int dy/\sqrt{y^2 + h^2} = \sinh^{-1}(y/h)$, where $\sinh^{-1}$ is the function inverse to sinh and h is a constant.)

(15 marks)

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } \; m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$