



SCHOOL OF MATHEMATICS AND STATISTICS

Spring semester
2010-2011

Applied Differential Equations

2 hours

Attempt all FOUR questions.

1 (i) The Euler 2 method, applied to the equation $dy/dx = f(x, y)$, is defined as:

$$\begin{aligned}k_1 &= hf(x_n, y_n) , \\k_2 &= hf(x_n + h, y_n + k_1) , \\y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2) .\end{aligned}$$

Given the ordinary differential equation with the initial condition:

$$\frac{dy}{dx} = 2x + y^2 - 2; \quad y(0) = 1, \quad (1)$$

use the Euler 2 method to find the approximate solution at $x = 0.2$ using step-size $h = 0.1$. Work throughout correct to four decimal places. (9 marks)

(ii) The Euler 2 method is used to solve equation (1) as far as $x = 2$, with two different step sizes. The results are

h	$y(2)$
0.05	2.86233
0.2	2.66393

Use these data to estimate the step size, h , required to ensure that the absolute value of the global discretization error in $y(2)$ is smaller than 5×10^{-3} . You may assume that the Euler 2 is a second order method. (7 marks)

(iii) Apply the Euler 2 method to the test equation $y'(x) = \lambda y(x)$, where λ is a constant, hence show that $y_{n+1} = R(\bar{h})y_n$, where $\bar{h} = \lambda h$ and $R(\bar{h})$ is a second order polynomial in \bar{h} . Find the expression for $R(\bar{h})$, hence show that the interval of absolute stability for the Euler 2 method is $(-2, 0)$. (9 marks)

- 2 (i) Determine the convergence, or the lack of it, of the linear multi-step method:

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}),$$

and the single-step method:

$$k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right),$$

$$y_{n+1} = y_n + \frac{2}{3}k_1 + \frac{1}{3}k_2,$$

where, in both cases, $f_n = f(x_n, y_n)$. State your arguments. **(19 marks)**

- (ii) The following table contains the grid-point values of the two solutions, $Y_1(x)$ and $Y_2(x)$, of a linear differential equation $d^2y/dx^2 = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method.

x	2.0	3.0	4.0
$Y_1(x)$	1.36831	2.12856	3.41281
$Y_2(x)$	2.15473	3.35557	5.85697

$Y_1(x)$ was determined using the initial conditions $y(1) = 1$, $y'(1) = 0$, and $Y_2(x)$ was obtained using $y(1) = 1$, $y'(1) = 0.5$. Form a linear combination of these two solutions which is the numerical solution to the equation $d^2y/dx^2 = f(x, y, y')$ with the boundary conditions

$$y(1) = 1, \quad y(4) = 2.$$

Calculate the value of this solution at each x -value given in the table. **(6 marks)**

- 3 (i) The function $u(x, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (0 \leq x \leq \ell, t \geq 0), \quad (2)$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(\ell, t) = 0,$$

and the initial condition

$$u(x, 0) = 2 \sin(\pi x/\ell) + \sin(2\pi x/\ell).$$

For a separable solution of the form $T(t)X(x)$, show that $X(x)$ satisfies the differential equation and the associated boundary condition:

$$X''(x) - \alpha X(x) = 0, \quad X(0) = X(\ell) = 0,$$

where α is a constant. Show that $T(t)$ satisfies the equation $T'(t) = k\alpha T(t)$. **(6 marks)**

- (ii) Assuming $\alpha = -s^2$ ($s > 0$), find the values of s such that there are non-trivial solutions for $X(x)$ and $T(t)$, and thus find $X(x)$ and $T(t)$. **(7 marks)**

(iii) The general solution of equation (2), with the given boundary conditions, can be written as

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} \exp \left(\frac{-n^2 \pi^2 k t}{\ell^2} \right),$$

where b_n are constants to be determined. Find the values of b_n such that $u(x, t)$ satisfies the given initial condition, hence write down the solution for $u(x, t)$. **(4 marks)**

- (iv) The function $w(x, t)$ satisfies the inhomogeneous heat equation

$$\frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2} + q, \quad (0 \leq x \leq \ell, t \geq 0)$$

with inhomogeneous boundary conditions

$$w(0, t) = T_0, \quad w(\ell, t) = T_1,$$

and initial condition

$$w(x, 0) = 2 \sin(\pi x/\ell) + \sin(2\pi x/\ell) + \frac{q}{2k} x(\ell - x) + \frac{T_1 - T_0}{\ell} x + T_0,$$

where q , T_0 and T_1 are constants. Find the solution $w(x, t)$ (Hint: You can use the solution you have obtained in part (iii) to avoid duplicate calculation). **(8 marks)**

- 4 (i) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \quad -\infty < x < \infty, \quad t \geq 0.$$

Given that the general solution to the wave equation is

$$u(x, t) = F(x + ct) + G(x - ct),$$

where $F(x)$ and $G(x)$ are two arbitrary functions, find the expressions for $F(x)$ and $G(x)$ such that $u(x, t)$ satisfies the initial conditions

$$u(x, 0) = \frac{1}{1 + x^2}, \quad \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} = 0.$$

Hence write down the solution $u(x, t)$. **(9 marks)**

- (ii) The function $u(x, y)$ satisfies the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the rectangular region $0 \leq x \leq a, 0 \leq y \leq b$, subject to boundary conditions:

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = -\sin(2\pi x/a). \quad (3)$$

Given that the solution is of the form $u(x, y) = -g(y) \sin(2\pi x/a)$ for some unknown function $g(y)$, find $g(y)$ and hence the solution $u(x, y)$. **(12 marks)**

- (iii) The function $w(x, y)$ satisfies the Laplace's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

in the rectangular region $0 \leq x \leq a, 0 \leq y \leq b$, subject to boundary conditions:

$$w(0, y) = 0, \quad w(a, y) = 2 \sin(\pi y/b), \quad w(x, 0) = 0, \quad w(x, b) = -\sin(2\pi x/a).$$

The function $w(x, y)$ can be found as the sum of two functions $u(x, y)$ and $v(x, y)$ where $u(x, y)$ and $v(x, y)$ both satisfy the Laplace's equation subject to suitable boundary conditions. Find the boundary conditions for u and v so that the equations for u and v can be solved using the usual method of separation of variables. Do **NOT** attempt to solve the equations for u and v .

(4 marks)

End of Question Paper