The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn 2010-2011

Introduction to Mathematical Finance and Time Series — 3 hours

Marks will be awarded for your best ***five*** *answers.*

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Consider the following two risk-free zero-coupon bonds with face value of £100:

Time to maturity (in years)	Bond price (in £)
1	97.53
3	90.03

Suppose that you are offered by a risk-free institution the opportunity to deposit or borrow £1,000,000 in one year for a period of two years earning an interest rate of 3%. Describe in detail an arbitrage opportunity available to you. ***(12 marks)***

(ii) Consider an American call option on shares with strike price X and expiring in T years. Assume that owning the shares for the next T years does not entitle the owner to dividends. Let $0 \leq \tau < T$, and let S_τ denote the share price at time τ.

(a) What is the payoff obtained by exercising the option at time τ? ***(2 marks)***

(b) Explain how (a) implies that the option should not be exercised at time τ. ***(4 marks)***

(c) Deduce that the price of the option equals that of a European call option with the same underlying asset, strike price and expiration date. ***(2 marks)***

2 (i) Consider a derivative on a stock which entitles the holder to one payoff at time T; the amount of this payoff is £1 if the stock price S_T at time T is at most a, for some positive number a, and zero otherwise. Let S be the price of the stock and assume, as usual, that S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and $\sigma > 0$ and where B is a Brownian motion. Assume further that all interest rates are constant and equal to r.

(a) Use Ito's Lemma to show that $\log S$ follows the process

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$ ***(6 marks)***

(b) Find an expression for the probability *in a risk-neutral world* of the event $S_T \leq a$. ***(6 marks)***

(c) Apply a risk-neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative at time t equals

$$e^{-r(T-t)}\Phi\left(\frac{\log(a/S_t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. ***(3 marks)***

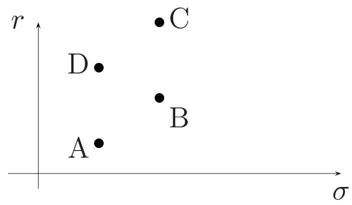
(ii) We are given the fact that the function $v(S,t) = e^{(4r+\sigma^2)(t-T)/8}\sqrt{S}$ is a solution of the Black-Scholes PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

Consider a derivative on a certain stock (whose price S, as always, follows the process $dS = S\mu dt + S\sigma dB$) which provides a single payoff at time $T > 0$ amounting to $\sqrt{S_T}$. Find the value of the derivative at time $0 \leq t \leq T$. ***(5 marks)***

3 (i) (a) Explain the term *efficient frontier* in the context of portfolio theory. ***(2 marks)***

(b) Which portfolios among A,B,C and D pictured below on the $\sigma - r$ plane (where σ denotes standard deviation of returns and r denotes expected returns) could be on an efficient frontier? Explain your answer in detail. ***(2 marks)***



(ii) Consider a world where there are only two risky investments: *Guns R Us* and *Yummy Butter* stocks.

	Number of shares	Price per share	Expected return	Standard deviation of return
Guns R Us	5,000,000	£1	20%	30%
Yummy Butter	3,000,000	£5	10%	10%

The correlation between the returns of these two stocks is 1/10.

(a) What is the market portfolio? ***(2 marks)***

(b) What are the expected return and standard deviation of returns of the market portfolio? ***(3 marks)***

(c) Find the beta-coefficient of *Guns R Us*. ***(3 marks)***

(d) What is the risk-free return in this world? ***(3 marks)***

(e) Assume that risk-free deposits are available. Of all portfolios consisting of cash-deposits and the two risky investments with expected returns of 10%, which one has the lowest standard deviation of returns. ***(5 marks)***

4 (i) (a) In the context of descriptive analysis of time series x_t, briefly explain why a moving average for even span s is *not* defined as

$$\frac{1}{s}(x_{t-s/2} + x_{t-s/2-1} + \cdots + x_{t-1} + x_t + x_{t+1} + \cdots + x_{t+s/2-1}).$$

(1 mark)

(b) Consider the time series with values

$$x_1 = 5, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 5, \quad x_5 = 7, \quad x_6 = 6, \quad x_7 = 3.$$

Using the *correct* definition of the even-span moving average, calculate moving averages of span 4, for the values x_3, x_4 and x_5.

(3 marks)

(ii) A time series of length 70 gave values for the sample autocorrelation function (ACF), denoted by r_h and values for the partial ACF, denoted by a_h, according to the table below.

Lag h	1	2	3	4
r_h	0.58	0.43	0.37	0.22
a_h	$*$	$*$	0.19	0.21

(a) Using this table, find the values of a_1 and a_2, indicated in the table by stars. ***(4 marks)***

(b) Test whether this time series is consistent with a white noise process, a moving average model and an autoregressive model. ***(10 marks)***

(c) Suggest a model which you would expect to fit well to this time series data. ***(2 marks)***

5 Consider the time series model

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t + \frac{1}{3}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2}, \qquad (1)$$

where ϵ_t is a white noise process with variance 3, i.e. $\epsilon_t \sim WN(0,3)$.

(i) Give the abbreviated name of the model for X_t. ***(1 mark)***

(ii) Write down model (1) in compact form, using the backward shift operator B. ***(2 marks)***

(iii) Show that model (1) is causal and invertible. ***(5 marks)***

(iv) Find the variance of X_t. ***(12 marks)***

6 Consider the trend dynamic linear model, given by equations

$$X_t = [1, 0] \begin{bmatrix} \theta_{1t} \\ \theta_{2t} \end{bmatrix} + \epsilon_t = \boldsymbol{F}^T \boldsymbol{\theta}_t + \epsilon_t, \tag{2}$$

$$\boldsymbol{\theta}_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t = \boldsymbol{G}\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \tag{3}$$

where $\boldsymbol{\theta}_t = [\theta_{1t}, \theta_{2t}]^T$ is a state vector, ϵ_t follows a normal distribution with zero mean and variance 50, and $\boldsymbol{\omega}_t$ follows a bivariate normal distribution with zero mean vector and covariance matrix

$$\boldsymbol{W} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

written as

$$\boldsymbol{\omega}_t \sim N_2 \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \right\}.$$

It is also assumed that ϵ_t and $\boldsymbol{\omega}_t$ are mutually and individually independent, and they are independent of the initial state $\boldsymbol{\theta}_0$. Suppose that $x_1, x_2, \ldots, x_n$ values of the time series are observed and that the posterior distribution of $\boldsymbol{\theta}_n$, given information $x^n = (x_1, \ldots, x_n)$ is given by

$$\boldsymbol{\theta}_n | x^n \sim N_2 \left\{ \begin{bmatrix} 250 \\ 100 \end{bmatrix}, \begin{bmatrix} 10 & 0 \\ 0 & 33 \end{bmatrix} \right\}.$$

For some positive integer $k > 0$, define the new time series

$$S_n = X_{n+1} + X_{n+2} + \cdots + X_{n+k}.$$

(i) Show that the k-step forecast function of $\{X_t\}$ is $\widehat{X}_{n+k} = E(X_{n+k}|x^n) = 100k + 250.$ ***(4 marks)***

(ii) Find the posterior mean of S_n, given x^n, for $k = 2$. ***(2 marks)***

(iii) For $k = 2$, show that, given x^n, the covariance of X_{n+1} and X_{n+2} is 96, and hence calculate the posterior variance of S_n, given x^n. ***(13 marks)***

(iv) Derive the posterior distribution of S_n, given x^n, for $k = 2$. ***(1 mark)***

End of Question Paper