Data provided: Formula sheet

MAS165

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-2012**

Mathematics for Physicists **2 hours**

This paper comprises TWO sections. You should attempt ALL questions on this paper.

Section A

A1 Given the point $A = (1, 2, -1)$ and the line $\mathbf{r} = (1, 0, 1) + \lambda(3, 0, -1)$.

(i) Find the shortest distance from A to the line. *(5 marks)*

(ii) Find the equation of the plane passing through A whose normal is parallel to $\mathbf{r}$. *(4 marks)*

(iii) Find the perpendicular distance of the plane from the origin (0,0,0). *(3 marks)*

A2 (i) Show that $(\mathbf{c} \times (\mathbf{b} \times \mathbf{c})) \times \mathbf{c} = c^2 \mathbf{b} \times \mathbf{c}$, where c is the magnitude of the vector $\mathbf{c}$. *(3 marks)*

(ii) If $\mathbf{r} = (1, 1, 1)$, $\mathbf{s} = (2, 0, \lambda)$ and $\mathbf{t} = (0, 1, 3)$, find $\mathbf{r} \times \mathbf{s}$ and $\mathbf{s} \times \mathbf{t}$ and hence show that

$$\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{t}$$

is true for all values of λ. *(6 marks)*

A3 Gauss' theorem may be written:

$$\oint_S \mathbf{G} \cdot \hat{\mathbf{n}} dS = \int_V (\nabla \cdot \mathbf{G}) dV \ .$$

Indicate whether the following statements about Gauss' theorem, as expressed here, are true or false

(i) The term $(\nabla \cdot \mathbf{G})$ is the curl of the vector field $\mathbf{G}$.

(ii) The integral on the right hand side is a volume integral.

(iii) The theorem relates the behaviour of the vector field $\mathbf{G}$ inside the volume V to the behaviour of $\mathbf{G}$ in the surface S that bounds V.

(iv) $\oint_S dS$ is a surface integral, over the surface S.

(4 marks)

Section B

B1 (i) The electric potential $V(x, y, z)$ for a point charge Q situated at the origin may be expressed in appropriate units as

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}},$$

where ϵ_0 is a positive constant. Verify that V satisfies Laplace's equation.

(10 marks)

(ii) The parametric equation of a right circular helix maybe described as

$$\mathbf{r} = (a \cos\theta, a \sin\theta, b\ \theta),$$

where a and b are positive constants. Find the unit vector along the tangent to the curve. ***(4 marks)***

(iii) Evaluate the integral

$$I = \int \mathbf{r} \cdot d\mathbf{r},$$

where $\mathbf{r} = (a \cos\theta, a \sin\theta, b\ \theta)$, a and b are positive constants and $0 \le \theta \le 4\pi$.

(11 marks)

B2 (i) The surface density of a plane lamina is given by

$$\sigma(x,y) = \sigma_0 \frac{x^2 + y^3}{a},$$

where σ_0 and a are constant. Find the mass of the lamina by evaluating the integral

$$M = \int_{\mathcal{R}} \sigma(x,y) dx dy,$$

where $\mathcal{R}$ is the region bounded by the y-axis (i.e. $x = 0$), the line $y = 2a$ and the line $y = x$. Integrate first over x and then over y. Evaluate then the same integral again by reversing the order of integration (i.e. first integrating over y and then over x). ***(15 marks)***

(ii) Consider a surface specified by the equation

$$x^3 + y^3 + z^3 = 3.$$

Find the unit normal vector to the surface at the point $(1, 1, 1)$. Find furthermore the equation for the tangent plane at that point. ***(5 marks)***

(iii) Let $\phi(x, y, z)$ be a generic scalar function. Show by explicit calculation that the gradient of ϕ obeys

$$\nabla \times (\nabla \phi) = 0.$$

(5 marks)

B3 (i) A scalar function is given in spherical polar coordinates by

$$V = \frac{A}{r} \sin^2\theta \ \sin\phi,$$

where A is a positive number. Calculate the gradient of V. ***(5 marks)***

(ii) Given that

$$\frac{\partial f}{\partial x} = 3x^2y^2 + y\cos x$$

and

$$\frac{\partial f}{\partial y} = 2x^3y - 2y\sin(y^2) + e^y + \sin x,$$

find $f(x, y)$. ***(7 marks)***

(iii) Find the work done in a moving particle P along a curve C specified by

$$\mathbf{r} = (1, 2, -1) + (t, t^2, 2t), 0 \le t \le 1$$

against a force specified by $\mathbf{F}$ = 2xy $\mathbf{i}$ + (x+y)$\mathbf{k}$. ***(13 marks)***

End of Question Paper

Mathematical Formulae:

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$
$$dV = r^2\sin\theta dr d\theta d\phi \quad \text{(Element of volume)}$$

In the following $\mathbf{F} = F_1\hat{\mathbf{r}} + F_2\hat{\boldsymbol{\theta}} + F_3\hat{\boldsymbol{\phi}}$ (note that $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors):

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_1\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_2\right) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

and

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}.$$

Let f be a scalar function, then the gradient is given by

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}.$$

Plane Polar Coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dA = dxdy = rdrd\theta$$

Vector Calculus:

$$\begin{aligned}
\nabla\phi &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \\
\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \\
\nabla^2\phi &= \nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
\nabla\times(\nabla\phi) &= 0 \\
\nabla\cdot(\nabla\times\mathbf{A}) &= 0 \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Vectors:

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y + A_z B_z \\
\mathbf{A} \times \mathbf{B} &= (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k} \\
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}
$$

Trigonometry:

$$
\begin{aligned}
\sin(\phi \pm \theta) &= \sin\phi\cos\theta \pm \cos\phi\sin\theta \\
\cos(\phi \pm \theta) &= \cos\phi\cos\theta \mp \sin\phi\sin\theta \\
\tan(\theta \pm \phi) &= \frac{\tan\phi \pm \tan\theta}{1 \mp \tan\phi\tan\theta} \\
\sin(2\phi) &= 2\sin\phi\cos\phi \\
\cos(2\phi) &= 2\cos^2\phi - 1 = 1 - 2\sin^2\phi \\
\sin\phi + \sin\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\sin\phi - \sin\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi + \cos\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi - \cos\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)
\end{aligned}
$$