The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011-2012**

Mathematics III (Control) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1. (i) Reduce the following matrix to Hermite form, specifying each row operation using any appropriate notation:

$$\begin{pmatrix} 2 & 6 & 1 & 8 & -10 \\ 1 & 3 & 2 & 1 & 1 \\ 3 & 9 & -1 & 17 & -22 \end{pmatrix}.$$

(12 marks)

(ii) Using the result of part (i), or otherwise, show that the planes

$$\begin{aligned} 2x+6y+z &= 8, \\ x+3y+2z &= 1, \\ 3x+9y-z &= 17 \end{aligned}$$

intersect on a common line, and find its parametric equation.

(5 marks)

(iii) Discuss the intersection of the three planes

$$\begin{aligned} 2x+6y+z &= -10, \\ x+3y+2z &= 1, \\ 3x+9y-z &= -22. \end{aligned}$$

(3 marks)

(iv) Solve the equations

$$\begin{aligned} 2x+y-10z &= 8, \\ x+2y+z &= 1, \\ 3x-y-22z &= 17. \end{aligned}$$

(5 marks)

2. (i) State which of the following are linear transformations, and state their domains and codomains: (You may assume that all variables are real numbers)

$$T_1(x, y, z) = (3x - y + 2z,\ x - 2y + 3z)$$
$$T_2(x, y) = (x^2 + y^2,\ 2xy)$$
$$T_3(x, y) = (2x + 3y,\ x - 5y,\ 2y)$$

For any which are *not* linear transformations show that at least one of the properties of linear transformations is not satisfied.

(9 marks)

(ii) For each of the linear transformations in part (i) write down the matrix which represents the transformation with respect to the standard bases for the domain and codomain. ***(4 marks)***

(iii) Find new bases for the domain and codomain of T_1 (of part (i) above) so that the matrix representing the transformation with respect to these new bases is in Smith form. ***(8 marks)***

(iv) Show that the image of T_3 (of part (i) above) is a plane and find the Cartesian equation of this plane. ***(4 marks)***

3. A real 8×8 matrix A has minimum polynomial $(x^4 - 2x^2 + 1)(x^2 - 4x + 5)$.
The trace of A is 6.
The geometric multiplicity of each eigenvalue is an odd number.

(i) Find all the (real and complex) eigenvalues of A. ***(5 marks)***

(ii) Find the geometric multiplicities of the real eigenvalues. ***(4 marks)***

(iii) Find the determinant of A. ***(2 marks)***

(iv) Find the invariant factors and write down the rational form of A. ***(7 marks)***

(v) Find the elementary divisors and write down a Jordan form of A. ***(7 marks)***

4. Let A be the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -7 & -5 \end{pmatrix}.$$

(i) Show that $\begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$ is an eigenvector of A and find the associated eigenvalue.

(3 marks)

(ii) Find all the eigenvalues of A and state the algebraic multiplicity of each.

(4 marks)

(iii) Find the eigenspace associated with each eigenvalue and hence state the geometric multiplicity of each eigenvalue.

(4 marks)

(iv) Write down the Jordan form J of A in the form

$$J = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

(2 marks)

(v) Show that $J = P^{-1}AP$, where

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 9 & 1 & -2 \end{pmatrix}.$$

(4 marks)

(vi) Hence or otherwise find the general solution of the differential equation

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + 5\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 7\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0.$$

(8 marks)

5. (i) Give a matrix representation of the quadratic form

$$Q(x,y,z) = 3x^2 + 2y^2 + z^2 - 4xy + 4yz$$

(4 marks)

(ii) Find the eigenvalues and normalised eigenvectors for the matrix obtained in part (i). ***(16 marks)***

(iii) Write $Q(x,y,z)$ as a sum or difference of squares and state whether it is positive definite, negative definite or indefinite. ***(3 marks)***

(iv) Give a brief description of the surface $Q(x,y,z) = 10$. ***(2 marks)***

6. (i) Find the stationary points of the function

$$z = f(x,y) = 2x^3 + 6xy^2 - 24xy + 18x$$

and investigate their nature.

(15 marks)

(ii) Use the method of Lagrange multipliers to find the maximum value of the function $f(x,y) = xy$ on the ellipse $9x^2 + 4y^2 = 72$ and give the points at which this maximum value is attained. ***(10 marks)***

End of Question Paper