SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester
2011–2012

MAS271 Methods for differential Equations

2 hours

Answer all four questions.

1 (i) Solve the equation

\[(1 - t^2)x'' - 6tx' - 4x = 0\]

equation near the ordinary point \(t = 0\). Give the general solution.

Using the ratio test, find the radius of convergence.

\((15 \text{ marks})\)

(ii) Do the following systems have equilibrium points? If so, find the equilibrium points and investigate the nature of the stability of the system at these points.

(a) \(\dot{x} = x(2 - x), \quad \dot{y} = -y + x\);

(b) \(\dot{x} = \frac{1}{y}, \quad \dot{y} = \frac{2}{x}\).

\((10 \text{ marks})\)

2 (i) Let

\[V = y^2 + \omega^2 x^2\]

and

\[\dot{x} = y, \quad \dot{y} = -cy - \omega^2 x \quad (\text{where } c > 0).\]

Deduce whether \(V\) is a weak or a strong Liapunov function and using this, comment on the stability of the point \((0,0)\).

\((6 \text{ marks})\)
(ii) Show that the first integral of the differential equation
\[ \ddot{x} + 4x = 0 \]
is
\[ 4x^2 + y^2 = \text{constant}, \]
where \( y = \dot{x} \).
Sketch these trajectories in the \( xy \)-plane, indicating the direction of time, \( t \), increasing.
What can you conclude about the nature of the equilibrium point \((0,0)\)? Justify your answer. \( (8 \text{ marks}) \)

(iii) Give general solution for \( x > 0 \):
(i) \( x^2 \ddot{y} + 5xy' + 4y = 0, \)
(ii) \( x^2 \ddot{y} + xy'' + y = 0. \) \( (11 \text{ marks}) \)

3 (i) Find the power series solution of
\[ x'' - tx' + x = 0 \]
with \( x(0) = 1 \) and \( x'(0) = 0. \) \( (7 \text{ marks}) \)

(ii) The system of three equations is given by
\[ (x, y, z)' = (4x - y, \ 3x + y - z, \ x + z). \]
express it in the vector-matrix form. \( (3 \text{ marks}) \)

(iii) Deduce whether the following Liapunov functions are positive definite or negative definite:
(a) \( 4x^2 + 3xy + 2y^2, \)
(b) \( -3x^2 - 4xy - y^2. \) \( (6 \text{ marks}) \)

(iv) Find and classify the equilibrium points of the system
\[ \dot{x} = x(\mu - x), \]
where \( \mu \) is a control parameter. Sketch the stable and unstable regions by using solid and dashed lines respectively. Explain what is meant by a transcritical bifurcation. \( (9 \text{ marks}) \)
4 (i) Show that the substitution \( z(x) = y(x)\sqrt{x} \) renders Bessel’s equation
\[
x^2y'' + xy' + (x^2 - \nu^2)y = 0; \quad \nu \geq 0,
\]
in the form
\[
z'' + \left( 1 + \frac{1 - 4\nu^2}{4x^2} \right) z = 0,
\]
where \( \nu \) is a constant. \((8\text{ marks})\)

For \( x \gg 1 \), the above equation can be approximated as
\[
z'' + z = 0.
\]
Write down the general solution. \((3\text{ marks})\)

By substituting \( y(x) = z(x)/\sqrt{x} \), write the general solution for \( y \). \((2\text{ marks})\)

(ii) For the following equations find the equilibrium points, investigate the nature of stability at these points and sketch the phase portrait for the stable equilibrium point:
\[
\dot{x} = 3x - x^2 - 2xy, \quad \dot{y} = -y + xy
\]
\((12\text{ marks})\)

End of Question Paper