The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-12**

Differential Equations: Case Studies in Applied Mathematics **Two hours**

Marks will be awarded for your best FOUR answers

A list of basic formulae and theorems for use as necessary is provided on the final sheet of the exam paper.

1 (i) Umberto D'Ancona studied the variations in population density of interacting fish species in the Mediterranean from data obtained during and after World War 1. In particular, the data gave the percentage-of-total catch of selachians (sharks, skate, etc). These figures are reproduced in the table below:

1914	1915	1916	1917	1918	1919	1920	1921	1922
11.9%	21.4%	22.1%	21.2%	36.4%	27.3%	16.0%	15.9%	14.8%

The data clearly show a marked year-on-year increase in this percentage until the final year of the war, 1918.

(a) What did D'Ancona reason to be the fundamental cause of this year-on-year increase? ***(2 marks)***

(b) What did his reasoning leave unexplained? ***(2 marks)***

(ii) Volterra's two-species model for food-fish and selachians, but *excluding the effects of fishing* is given by

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + kxy,$$

where the parameters a, b, c and k are all positive. State which equation describes which population and explain the significance of each of the right-hand-side terms in each equation. ***(6 marks)***

(iii) Identify the critical points (x^*, y^*) and, by performing a linear stability analysis, verify that the solutions of the linearised equations in the region of the non-trivial critical point are periodic. ***(8 marks)***

(iv) Remembering that the average value of a function $f(t)$ over the range $0 \leq t \leq T$ is given by

$$\bar{f} = \frac{1}{T}\int_0^T f(t)dt,$$

show that the average values of x and y over one whole period, T say, are given by the co-ordinates of the non-trivial critical point. ***(7 marks)***

2 (i) Consider the system of equations

$$\dot{x} = y(1 + x^2 + y^2),$$

$$\dot{y} = -x(1 + x^2 + y^2).$$

(a) Form $x\dot{x} + y\dot{y}$ and hence derive the solution in the form $x^2 + y^2 = r_0^2$, where r_0 is a constant. ***(5 marks)***

(b) By making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$x\dot{y} - y\dot{x} = r^2\dot{\theta}.$$

Hence show that

$$\dot{\theta} = -(1 + r^2).$$

(5 marks)

(c) For a given orbit with $r = r_0$, determine the period of the solution. ***(5 marks)***

(ii) Use Theorem 1 from the Formula Sheet to show that, for each of the following systems, there are no periodic solutions:

(a)

$$\dot{x} = 3x + 2y + x^5 - y^5, \quad \dot{y} = -x + 2y + x^3 + y^5,$$

(5 marks)

(b)

$$\dot{x} = y - x^3y^2, \quad \dot{y} = -ye^x + x^2.$$

(5 marks)

3 (i) (a) Suppose that a particle moves in finite steps Δx, each taking a finite time Δt, and suppose further that $p(x,t)$ is the probability that a particle released at $t=0$ reaches coordinate position x in time t. If (α,β) are the probabilities that a particle will move to the left or the right respectively, then write down an expression for $p(x,t)$ in terms of the probabilities of the possible states at $t-\Delta t$. ***(3 marks)***

(b) Hence, for the case of the isotropic walk ($\alpha=\beta$) in the limits of $\Delta x\to 0$ and $\Delta t\to 0$, show how the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2},$$

for a particle concentration $u(x,t)\equiv Qp(x,t)$, where $Q>>1$, and diffusion coefficient D, can be derived. You are given that $\alpha+\beta=1$. State clearly any assumptions you make about the limiting behaviour of Δx and Δt. ***(10 marks)***

(ii) (a) The one-dimensional heat conduction equation can be written in the form

$$\frac{\partial \theta}{\partial t}=D\frac{\partial^2 \theta}{\partial x^2},$$

where D is the conduction coefficient.

The maximum daily variation in temperature at the soil surface in a particular geographical location is found to be 30°C. The variation can be reasonably described by a function of the general form

$$\theta(0,t)=10+A(0)\cos\omega t,$$

where t is measured in hours. Determine the constants $A(0)$ and ω. ***(2 marks)***

(b) Hence, by assuming a solution to the heat conduction equation of the form

$$\theta(x,t)=10+A(x)\cos(\omega t-Cx)$$

where x is the depth below the surface and C is a constant, show that $A(x)\propto e^{-\sqrt{\frac{\omega}{2D}}x}$. You may assume that $C>0$. ***(10 marks)***

4 (i) (a) A travelling wave $u(x \pm ct)$ can have one of three distinct forms. Describe, briefly, what these are. ***(3 marks)***

(b) Show that the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

where D is the diffusion coefficient, allows solutions of the form $u(x \pm ct)$. (*Hint:* make the substitution $z = (x \pm ct)$.) ***(7 marks)***

Explain why these solutions *cannot* be considered as travelling waves. Briefly, and in words, explain what additional properties a diffusing system requires before it can have travelling wave solutions.

(3 marks)

 Question 4 continued on next page

4 (continued)

(ii) A particular chemical reaction, which occurs on the surface of an expanding spherical bubble, can be described in the form of the diffusion equation modified with an interaction term. After suitable coordinate transformations, the full model is given by

$$\frac{\partial u}{\partial t} = D\nabla^2 u + ku(0.5-u)(1-u)$$

where, assuming that angular variations in the reaction can be ignored, $\nabla^2 u$, the Laplacian term, is given in spherical polar coordinates, (r, θ, ϕ), by

$$\nabla^2 u \equiv \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial u}{\partial r}\right)\right].$$

k is a constant.

(a) Derive the model equation in the form

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{D}{k}\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right] + u(0.5-u)(1-u).$$

(5 marks)

(b) Rescale the model equation using $T = kt$ and $R^2 = \dfrac{kr^2}{D}$.

(2 marks)

(c) Use the transformation $u = u(R - cT)$, $z = R - cT$ to show that in the case of large R (that is, in the *far field*) the model equation becomes

$$\frac{d^2u}{dz^2} + c\frac{du}{dz} + u(0.5-u)(1-u) = 0.$$

(5 marks)

5 (i) Maupertuis is quoted to have said (in translation)

> If there occurs some change in nature, the amount of action necessary for this change must be as small as possible.

He did not define what he meant here by action but we can reasonably suppose that he had some intuitive notion of it. In modern science, we define action as the product of energy and time so that, for example, we could say that it requires one unit of action for a man to walk one mile in one hour. Given this, how many units of action are required for a man to walk two miles in one hour? State your reasoning clearly. ***(2 marks)***

(ii) The mathematicians Euler and Lagrange laid down the basis of that part of mathematics known as the calculus of variations and their work is widely supposed to have been motivated by the work of John Bernoulli (and Newton in the same year, 1697) in solving the long-standing brachystochrone problem which can be stated in the following terms:

> A curved wire connects two fixed points A and B which are at different heights but are not in vertical alignment. A frictionless bead is allowed to descend the wire under gravity alone. Find the curve of the wire which minimizes the time of descent of the bead.

Show that this problem can be expressed as that of finding the plane curve $y(x)$ such that

$$T = \int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{2g(y_A - y)}}dx$$

is minimized. Here, g is the usual acceleration due to gravity, y_A is the initial $y-$coordinate of the bead and it is assumed that the initial velocity of the bead is zero. ***(6 marks)***

 Question 5 continued on next page

5 (continued)

(iii) (a) Suppose that $x^*(t)$ is a minimising curve for the functional

$$J[x] = \int_{t_o}^{t_1} f(x, x', t)dt$$

where $x(t_0)$ and $x(t_1)$ are fixed. Assuming, for simplicity, that the admissible curves belong to $\mathcal{C}_2$, derive the first necessary condition

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 0$$

that the function $x^*(t)$ must satisfy. You may assume the result that if $g(t)$ is continuous in $[a, b]$ then $\int_a^b \eta(t)g(t)dt = 0$ for all $\eta(t)$ implies that $g(t) = 0$ for all $t \in [a, b]$. ***(8 marks)***

(b) Given that, if t does not appear explicitly in the functional so that $f \equiv f(x, x')$, then the first necessary condition for a minimising curve can be expressed as

$$f - x'\frac{\partial f}{\partial x'} = \text{constant},$$

find the extremal curve for the fixed end point problem

$$h[x] = \int_0^1 \frac{(1 + {x'}^2)^{\frac{1}{2}}}{x} dt$$

with $x(0) = 0$ and $x(1) = \sqrt{5}$. ***(9 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$