The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2011–2012

Rings and Groups — **2 hours**

Answer ***all four questions****.*
You should justify your answers carefully unless the question states otherwise.

1 (i) Write down all the units in the ring $\mathbb{Z}_{24}$, justifying your answer. ***(4 marks)***

(ii) Let R be any ring. Prove that the units of R form a multiplicative group $\mathcal{U}(R)$. ***(5 marks)***

(iii) What does it mean for a group to be cyclic? Is the group $\mathcal{U}(\mathbb{Z}_{24})$ cyclic? ***(4 marks)***

(iv) Give an example of a ring R such that $\mathcal{U}(R)$ is the cyclic group of order 2. ***(2 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

2 (i) Which of the following rings are integral domains? Justify your answers briefly.

(a) $\mathbb{C}$

(b) $\mathbb{Z}_{31}[x]$

(c) $\mathbf{Mat}_2(\mathbb{R})$ ***(7 marks)***

(ii) What does it mean for a ring R to be a unique factorisation domain? ***(4 marks)***

(iii) Consider the following two factorisations of 20 in $\mathbb{Z}[i]$:

$$20 = (4 + 2i)(4 - 2i)$$

$$20 = 4.5$$

Do these two factorisations show that $\mathbb{Z}[i]$ is not a unique factorisation domain? ***(4 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

3 (i) What is meant by the class equation of a group G? ***(2 marks)***

(ii) Write down all possible cycle types in S_4, together with the number of elements in S_4 of each type. Hence write down the class equation for S_4. Justify your answers. ***(8 marks)***

(iii) Are the following subgroups of S_4 normal? Justify your answers.

(a) A_4

(b) $\langle(12)(34)\rangle$ ***(5 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

4 Write C_4 for the cyclic group of order 4 generated by a under multiplication. So C_4 has elements $1, a, a^2, a^3$ and a satisfies the equation $a^4 = 1$. Write $\mathbb{Z}$ for the group of integers under addition.

Let f be the map defined as follows

$$\begin{array}{cccc} f & : & \mathbb{Z} & \longrightarrow & C_4 \\ & & n & \mapsto & a^n. \end{array}$$

(i) Show that f is a group homomorphism. ***(2 marks)***

(ii) Let $\theta : G \longrightarrow H$ be a group homomorphism. Define the kernel and image of θ and state the First Isomorphism Theorem for groups. ***(4 marks)***

(iii) Find the kernel and image of f, justifying your answers. What can we conclude using the First Isomorphism Theorem for groups? ***(6 marks)***

(iv) Does the following map define a group homomorphism?

$$\begin{array}{ccccc} g & : & C_4 & \longrightarrow & \mathbb{Z} \text{ by} \\ & & a^0 & \mapsto & 0 \\ & & a^1 & \mapsto & 1 \\ & & a^2 & \mapsto & 2 \\ & & a^3 & \mapsto & 3 \end{array}$$

(3 marks)

Additional marks for rigour and presentation. ***(5 marks)***

End of Question Paper