Section A

A1 Use the binomial theorem to find the value of \( a \) for which \( \lim_{x \to 0} \frac{(8 - ax)^{1/3} - 2}{x} = 1 \).

A2 If \( f(x) = \frac{3x}{x + 2}, x \geq 0 \), what is the range of \( f(x) \)? Find \( f^{-1}(y) \).

A3 Find the first three nonzero terms in the Maclaurin expansion of \( f(x) = e^{-2x} \cos 3x \).

A4 Evaluate \( \lim_{x \to \pi/2} \frac{x \cos 3x}{2x - \pi} \).

A5 Find all the complex numbers \( z \) for which \( \text{Re}(z) + \text{Im}(z) = 0 \) and \( |z - 1 + i| = \sqrt{2} \).
If \( \mathbf{a} = (3, 1, -1) \) and \( \mathbf{b} = (1, -2, 1) \), show that \( \mathbf{a} \) is perpendicular to \( \mathbf{b} \) and find \( \mathbf{a} \times \mathbf{b} \).

Use integration by parts to find the indefinite integral

\[
\int x^n \ln(x) \, dx
\]

where \( n \neq -1 \).

Compute the definite integral

\[
\int_{-1}^{1} \sin^2 \left( \frac{\pi x}{2} \right) \, dx.
\]

Find the determinant of \( A^{-1}B \), where \( A \) and \( B \) are given by

\[
A = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Consider the homogeneous linear system of equations

\[
\begin{align*}
3x + y &= 0, \\
ax - 4y &= 0,
\end{align*}
\]

which has the trivial solution \( x = y = 0 \). Find the value of \( a \) for which the system has additional solutions.

Find the general solution of the following differential equation:

\[
\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0.
\]

Show that the general solution of the differential equation

\[
\frac{dy}{dx} = \frac{y}{1 - x^2},
\]

is given by \( y = A \sqrt{\frac{1 + x}{1 - x}} \), where \( A \) is a constant.
Section B

B1 Find the quadratic $Q(x)$ for which the polynomial $P(x) = x^3 - x^2 - x - 15$ can be written in the form

$$P(x) = (x - 3) Q(x).$$

Find the roots of $Q(x)$ and plot them on an Argand diagram. Find the constants $A$ and $B$ such that

$$Q(x) = (x - A)^2 + B.$$

Sketch the curve $y = Q(x)$.

B2 Determine the modulus and principal argument of the complex numbers $z_1 = -3 + 4i$ and $z_2 = -4 - 3i$.

By expressing $z_1$ and $z_2$ in exponential form, or otherwise, show that

$$\frac{z_1^3}{z_2^3} = \frac{1}{25} \bar{z}_1,$$

where $\bar{z}_1$ is the complex conjugate of $z_1$.

B3 Find the values of $x$ at which the function

$$f(x) = x^3 + 2x^2 - 4x + 1$$

has stationary points. Determine the nature of each of the stationary points and sketch the curve $y = f(x)$.

B4 A function $f(x,y)$ of two variables is given by

$$f(x,y) = \left(2x^5 - \frac{1}{x^5}\right) \cos 5y.$$

Calculate the first-order partial derivatives $f_x$ and $f_y$. Calculate the second-order partial derivatives $f_{xx}$ and $f_{yy}$ and demonstrate that

$$x^2 f_{xx} + xf_{x} + f_{yy} = 0.$$

B5 Let

$$I = \int \frac{dx}{x^2 + 2x + a}$$

where $a$ is a constant. Find the indefinite integral $I$ in the following cases,

(i) $a = 1$,

(ii) $a = 0$,

(iii) $a = 2$.
B6 Let

\[
A = \begin{bmatrix}
1 & 0 & 2 \\
0 & 3 & 1 \\
1 & -4 & 1
\end{bmatrix}.
\]

(a) Find $|A|$, the determinant of $A$.

(b) Find $A^{-1}$, the inverse of $A$.

(c) Use $A^{-1}$ to find $x$, $y$ and $z$ which are solutions to the linear equations

\[
\begin{align*}
x + 2z &= 5, \\
3y + z &= 11, \\
x - 4y + z &= -9.
\end{align*}
\]

B7 Let

\[
A = \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}.
\]

(a) Find the eigenvalues and normalized eigenvectors of $A$.

(b) Show that the eigenvectors are orthogonal.

(c) Four points $PQRS$ on a unit square are described by the column vectors

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1
\end{bmatrix},
\]

respectively. The transformed points $P'Q'R'S'$ are obtained by multiplying the column vectors by matrix $A$. Draw a sketch of the $xy$ plane showing the original points $PQRS$, the transformed points $P'Q'R'S'$, and the direction of the eigenvectors of $A$.

B8 Consider the following differential equation:

\[
y'' + 2y' + 2y = f(x)
\]

(a) Find the general solution in the case $f(x) = 0$.

(b) Find a particular solution for $f(x) = \cos x$.

(c) Find the solution for $f(x) = \cos x$ with initial conditions $y(0) = y'(0) = 0$.

End of Question Paper
These results may be quoted without proof unless proofs are asked for in the questions.

**Trigonometry**

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \cosec^2 x \\
2\sin^2 x &= 1 - \cos 2x \\
2\cos^2 x &= 1 + \cos 2x \\
2\sin x \cos x &= \sin 2x \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \\
a \cos x + b \sin x &= R \cos(x - \alpha) \\
& \quad \text{where} \quad R = \sqrt{a^2 + b^2} \\
& \quad \cos \alpha = \frac{a}{R} \quad \text{and} \quad \sin \alpha = \frac{b}{R} \\
2\cos x \cos y &= \cos(x + y) + \cos(x - y) \\
2\sin x \sin y &= \cos(x - y) - \cos(x + y) \\
2\sin x \cos y &= \sin(x + y) + \sin(x - y) \\
\cos x &= \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} \\
\sin x &= \frac{2\tan(x/2)}{1 + \tan^2(x/2)} \\
\tan x &= \frac{2\tan(x/2)}{1 - \tan^2(x/2)}
\end{align*}
\]

**Hyperbolic Functions**

\[
\begin{align*}
\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
\text{sech} x &= \frac{1}{\cosh x} \\
\cosh^2 x - \sinh^2 x &= 1 \\
2\cosh^2 x &= 1 + \cosh 2x \\
2\sinh^2 x &= \cosh 2x - 1 \\
2\sin x \cosh x &= \sinh 2x \\
\text{sech}^2 x &= 1 - \tanh^2 x \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \quad , \text{all } x \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \quad , \text{ } x \geq 1 \\
\tanh^{-1} x &= \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \quad , |x| < 1
\end{align*}
\]
Series

Sum of an arithmetic series:
\[
\frac{\text{first term} + \text{last term}}{2} \times (\text{number of terms})
\]

Sum of a geometric series: \[1 + x + x^2 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}\]

Binomial theorem: \[(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \binom{n}{r} x^r + \ldots\]

where \[\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\]

If \(n\) is a positive integer then the series terminates and the result is true for all \(x\), otherwise, the series is infinite and only converges for \(|x| < 1\).

\[
\begin{align*}
\sin x & = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x & = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x & = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x & = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x & = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
\ln(1 + x) & = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)
\end{align*}
\]
<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
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<tr>
<td>tan (x)</td>
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<td>cot (x)</td>
<td>- cosec^2 (x)</td>
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<td>x</td>
<td>&lt; 1)</td>
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</tbody>
</table>
Integration

In the following table the constants of integration have been omitted.

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)
\]

\[
\int e^x \, dx = e^x
\]

\[
\int \sin x \, dx = -\cos x
\]

\[
\int \sec^2 x \, dx = \tan x
\]

\[
\int \sinh x \, dx = \cosh x
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad (|x| < a)
\]

\[
\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}
\]

\[
\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad (= \tanh^{-1} \frac{x}{a} \text{ if } |x| < a)
\]

\[
\int \cosec x \, dx = \ln \tan \left( \frac{x}{2} \right) \quad \text{or} \quad \ln (\cosec x - \cot x)
\]

\[
\int \sec x \, dx = \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \quad \text{or} \quad \ln (\sec x + \tan x)
\]

\[
\int \cosech x \, dx = \ln \tanh \left( \frac{x}{2} \right)
\]
Integration by parts
\[ \int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx \]

Newton-Leibnitz formula
If \( F'(x) = f(x) \), then \( \int_a^b f(x) \, dx = F(b) - F(a) = F(x) \bigg|_a^b \)

Variable substitution in definite integral
If \( x = \varphi(t) \) is a monotonic function in the interval \([\alpha, \beta]\) and \( a = \varphi(\alpha), \, b = \varphi(\beta) \), then
\[ \int_a^b f(x) \, dx = \int_{\alpha}^{\beta} f(\varphi(t)) \varphi'(t) \, dt \]

Variable substitution for a rational function of \( \sin x \) and \( \cos x \)
Let \( t = \tan \left(\frac{x}{2}\right) \) then \( \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \) and \( \frac{dx}{dt} = \frac{2}{1+t^2} \).

Area of planar figure

The area of a planar figure bounded by the graph of a continuous positive function \( f(x) \), the \( x \)-axis, and the ordinates \( x = a \) and \( x = b \) is
\[ S = \int_a^b f(x) \, dx \]

Volume of solid of revolution

The volume of a solid of revolution, obtained by rotation of a planar figure bounded by the graph of a continuous positive function \( f(x) \), the \( x \)-axis, and the ordinates \( x = a \) and \( x = b \) through a complete revolution around the \( x \)-axis, is
\[ V = \pi \int_a^b f^2(x) \, dx \]