All questions are compulsory. The marks awarded to each question or section of question are shown in italics.

1. Find the constants $r$ and $\alpha$, with $r > 0$ and $-\pi < \alpha \leq \pi$, such that

$$\cos x + \sqrt{3} \sin x = r \cos(x - \alpha).$$

Hence find all solutions $x$ in $(-\pi, \pi]$ of

$$\cos x + \sqrt{3} \sin x = \sqrt{2}.$$  \hspace{1cm} (4 marks)

2. Solve the following for $x$:

$$2 \log_a x - \log_a(x-1) = \log_a(x-2).$$ \hspace{1cm} (6 marks)

3. Identify the number $d$ such that

$$\frac{3^9 \cdot 4^{9/2}}{6(3^4)^2} = 2^d$$ \hspace{1cm} (4 marks)

4. Simplify

$$(5x^2y^{\frac{3}{2}}z^{\frac{1}{2}})^2 \times (4x^4y^2z)^{-1/2}$$ \hspace{1cm} (4 marks)

5. Factorise $3x^2 + 14x + 8$. \hspace{1cm} (2 marks)
6 Verify the following identity

\[ 1 - \frac{\sin \theta \tan \theta}{1 + \sec \theta} = \cos \theta \]

(5 marks)

7 State the range of the function \( f(x) = x^2 - 1 \).

(2 marks)

8 State the domain of the function

\[
y(x) = \begin{cases} 
  x^2 + 1 & -1 \leq x \leq 2 \\
  3x & 2 < x \leq 6 \\
  2x + 1 & x > 6.
\end{cases}
\]

(2 marks)

9 For the circle \( x^2 + y^2 = 5 \), find the equation of the tangent which meets the circle at \((-2, 1)\).

Find where this tangent intersects the circle with centre \((-3, 9)\) and radius 5.

(7 marks)

10 If \( y = a \cos(\omega x) + b \sin(\omega x) \), find \( \frac{d^2y}{dx^2} \) in terms of \( y \).

(4 marks)

11 (i) Showing your working clearly, find the coefficient of \( x^3 \) in the expansion of \((1 + x)^{14}\).

(2 marks)

(ii) Use the binomial theorem to evaluate

\[
\lim_{x \to 0} \frac{x^2}{\sqrt{4 - x^2 - 2}}.
\]

(3 marks)

12 (i) Find the angle between the vectors \( \mathbf{a} = (1, 0, 3) \) and \( \mathbf{b} = (2, 1, -1) \), giving your answer in radians to two decimal places.

(3 marks)

(ii) A plane passes through the points \((1, 2, 1), (0, 1, 0)\) and \((-1, 3, 2)\). Find the Cartesian equation of the plane.

(6 marks)

13 Prove, from the definitions of \( \sinh x \) and \( \cosh x \), the identity

\[
\sinh^2 x + \cosh^2 x = \cosh 2x.
\]

(3 marks)
14 (i) If $y = \ln(\sinh x)$, show that

$$\frac{dy}{dx} = \coth x.$$ \hspace{1cm} (2 marks)

(ii) If $y = \frac{\sinh 2x}{\cosh x}$, show that

$$\frac{dy}{dx} = 2 \cosh x.$$ \hspace{1cm} (4 marks)

15 By making the substitution $x = a \sinh \theta$, where $a$ is a positive constant, show that

$$\int_0^a \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} 1.$$ \hspace{1cm} (6 marks)

16 Find the Maclaurin series for $\ln(1 - x)$, as far as the term in $x^4$. \hspace{1cm} (7 marks)

17 Complex numbers $z_1$ and $z_2$ are defined by

$$z_1 = 1 + i, \quad z_2 = 3 - i.$$ Find, in the form $a + bi$ where $a$ and $b$ are real,

(i) $z_1^4$ \hspace{1cm} (2 marks)

(ii) $\frac{z_2}{z_1 + 2z_2}$ \hspace{1cm} (3 marks)

18 A set of linear equations can be written as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 3 & 1 & 2 \\ 1 & 2 & -4 \end{pmatrix}.$$ Find the inverse, $A^{-1}$, of $A$, and use it to find the values of $x$, $y$ and $z$ which satisfy the equations. \hspace{1cm} (9 marks)

End of Question Paper
These results may be quoted without proof, unless proofs are asked for in the question.

**Trigonometry**

For any angles $A$ and $B$

\[
\sin^2 A + \cos^2 A = 1 \\
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \\
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \\
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \\
\sin 2A = 2 \sin A \cos A \\
\cos 2A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A
\]

**Coordinate Geometry**

The acute angle $\alpha$ between lines with gradients $m_1$ and $m_2$ satisfies

\[
\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad (m_1 m_2 \neq -1)
\]

while the lines are perpendicular if $m_1 m_2 = -1$.

The equation of a circle centre $(x_0, y_0)$ and radius $a$ is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

**Hyperbolic Functions**

\[
\cosh^2 x - \sinh^2 x = 1 \\
\sech^2 x + \tanh^2 x = 1 \\
\cosh^2 x + \sinh^2 x = \cosh 2x \\
2 \sinh x \cosh x = \sinh 2x \\
\cosh^2 x = (1 + \cosh 2x)/2 \\
\sinh^2 x = -(1 - \cosh 2x)/2
\]
**Differentiation**

<table>
<thead>
<tr>
<th>Function $(y)$</th>
<th>Derivative $(dy/dx)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\sin ax$</td>
<td>$a \cos ax$</td>
</tr>
<tr>
<td>$\cos ax$</td>
<td>$-a \sin ax$</td>
</tr>
<tr>
<td>$\tan ax$</td>
<td>$a \sec^2 ax$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$ae^{ax}$</td>
</tr>
<tr>
<td>$\ln(ax)$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\ln f(x)$</td>
<td>$\frac{f'(x)}{f(x)}$</td>
</tr>
<tr>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
</tr>
<tr>
<td>$\cosh x$</td>
<td>$\sinh x$</td>
</tr>
<tr>
<td>$\tanh x$</td>
<td>$\text{sech}^2 x$</td>
</tr>
<tr>
<td>$\sin^{-1} x$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\cos^{-1} x$</td>
<td>$-\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>$\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\sinh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2+1}}$</td>
</tr>
<tr>
<td>$\cosh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2-1}}$</td>
</tr>
<tr>
<td>$\tanh^{-1} x$</td>
<td>$\frac{1}{1-x^2}$</td>
</tr>
</tbody>
</table>

NB. It is assumed that $x$ takes only those values for which the functions are defined.
For $u$ and $v$ functions of $x$, and with $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}.$$

**Integration**

In the following table the constants of integration have been omitted.

<table>
<thead>
<tr>
<th>Function $f(x)$</th>
<th>Integral $\int f(x) , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1}$, $n \neq -1$</td>
</tr>
<tr>
<td>$ae^{ax}$</td>
<td>$e^{ax}$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$a \sin ax$</td>
<td>$- \cos ax$</td>
</tr>
<tr>
<td>$a \cos ax$</td>
<td>$\sin ax$</td>
</tr>
<tr>
<td>$a \tan ax$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$\frac{1}{a^2 + x^2}$</td>
<td>$\frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right)$</td>
</tr>
<tr>
<td>$\frac{1}{a^2 - x^2}$</td>
<td>$\frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{a^2 - x^2}}$</td>
<td>$\sin^{-1} \left( \frac{x}{a} \right)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{x^2 + a^2}}$</td>
<td>$\sinh^{-1} \left( \frac{x}{a} \right)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{x^2 - a^2}}$</td>
<td>$\cosh^{-1} \left( \frac{x}{a} \right)$</td>
</tr>
<tr>
<td>$f'(x) \over f(x)$</td>
<td>$\ln</td>
</tr>
</tbody>
</table>

6
**Integration by parts**

\[ \int uV \, dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx} \, dx \]

or

\[ \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx. \]

**Series**

**Binomial Theorem:**

\[ (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \binom{n}{r}x^r + \cdots \]

where \( \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \)

If \( n \) is a positive integer, the series terminates and is convergent for all \( x \).

If \( n \) is not a positive integer, the series is infinite and converges for \( |x| < 1 \).

**Taylor expansion of \( f(x) \) about \( x = a \) is**

\[ f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \]

**Maclaurin expansion of \( f(x) \) is**

\[ f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots \]

**Alternating Series Test**

The series \( a_1 - a_2 + a_3 - a_4 + \cdots \), where \( a_1, a_2, a_3, a_4, \ldots \) are all positive, converges if \( a_1 > a_2 > a_3 > \cdots \) and \( a_n \rightarrow 0 \) as \( n \rightarrow \infty \).

**Ratio Test**

If the series \( a_1 + a_2 + a_3 + a_4 + \cdots \) satisfies

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda, \]

then

1. if \( \lambda > 1 \), the series diverges,
2. if \( \lambda < 1 \), the series converges.
**Vectors**

If vectors \( \mathbf{a} \) and \( \mathbf{b} \) are given in Cartesian component form by \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), then

the scalar product \( \mathbf{a} \cdot \mathbf{b} \) is given by

\[
\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3
\]

and the vector product \( \mathbf{a} \times \mathbf{b} \) is given by

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 
\end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1). 
\]

If a plane passes through a point with position vector \( \mathbf{a} \), and is normal to the vector \( \mathbf{n} \), then the equation of the plane is

\[
\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},
\]

where \( \mathbf{r} = (x, y, z) \).