A1 A plane is given by the equation
\[4x + 5y + 7z = 21\]
and a line by the equation \( \mathbf{r} = (1, 2, 3) + \lambda (1, 2, -2) \), where \( \lambda \) is a real parameter.

(i) Show that the line does not intersect the plane. \((4 \text{ marks})\)

(ii) Therefore, calculate the distance of the line to the plane. \((4 \text{ marks})\)

(iii) Find the direction of the line of intersection of the two planes
\[x + 3y - z = 5\]
and
\[2(x - y) + 4z = 3.\] \((5 \text{ marks})\)

A2 Show that \( f(x, y) = e^{-x} \cos(y) - e^{-y} \cos(x) \) obeys the two dimensional Laplace equation:
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
\]

Show that the function \( g(x, y) = \sqrt{x^2 + y - xy} \) does not obey the two dimensional Laplace equation. \((9 \text{ marks})\)
Stokes’ theorem may be written:
\[ \oint_C \mathbf{G} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{G}) \cdot \hat{n} \, dS \]

Indicate whether the following statements about Stokes’ theorem, as expressed here, are true or false

(i) The term \((\nabla \times \mathbf{G})\) is the divergence of the vector field \(\mathbf{G}\).

(ii) \(\hat{n}\) is a unit vector parallel with the boundary \(C\).

(iii) \(\int_S dS\) is a surface integral, over the surface \(S\).

(3 marks)

Section B

B1 (i) Find the work done by a force \(\mathbf{F} = (x + yz)i + (y + xz)j + (z + xy)k\) in moving a particle from the origin \(O\) to the point \(A(1,1,1)\)

(a) along the curve \(x = t, y = t^2, z = t^3\),

(b) along the straight line \(OA\).

(12 marks)

(ii) A scalar function is given as
\[ \phi(x, y, z) = x^2 - y \sin(x - z). \]

(a) Calculate the gradient of \(\phi(x, y, z)\), i.e. calculate \(\nabla \phi\).

(3 marks)

(b) Using your result, calculate the divergence of \(\nabla \phi\).

(4 marks)

(c) By explicit calculation, show that \(\nabla \times \mathbf{V} = 0\).

(6 marks)
B2  (i) A vector field is given by

\[ \mathbf{V} = V_1 \hat{r} + V_2 \hat{\theta} + V_3 \hat{z} = r \hat{r} + (a + r^3) \hat{\theta} + b \ln(z) \hat{z} \]

in cylindrical polar coordinates, where \( a \) and \( b \) are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

\[ \nabla \cdot \mathbf{V} = \frac{1}{r} \frac{\partial}{\partial r} (rV_1) + \frac{1}{r} \frac{\partial}{\partial \theta} (V_2) + \frac{\partial}{\partial z} (V_3) \]

and

\[ \nabla \times \mathbf{V} = \frac{1}{r} \begin{vmatrix} \hat{r} & r \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_1 & rV_2 & V_3 \end{vmatrix} \]

respectively. Can \( \nabla \times \mathbf{V} \) be zero? \( \text{(10 marks)} \)

(ii) Sketch the region of integration represented by the repeated integral

\[ \int \int_R x^2 y \, dx \, dy \]

where \( R \) is the region such that \( x \geq 0 \), \( y \geq 0 \), and \( x^2 + y^2 \leq a^2 \). By transforming to plane polar coordinates, evaluate the integral. \( \text{(15 marks)} \)

B3  (i) Verify the divergence theorem

\[ \int_V (\nabla \cdot \mathbf{A}) \, dV = \oint_S \mathbf{A} \cdot \mathbf{n} \, dS, \]

for the vector field \( \mathbf{A} = (x, y, z) \) and \( S \) being the surface enclosing a cylinder of radius \( a \) (i.e. \( x^2 + y^2 = a^2 \)) and height \( h \). The bottom surface of the cylinder lies in the \( xy \)-plane \( (z = 0) \). Hint: split the surface integral into three parts. To find \( \mathbf{n} \) for the curved surface of the cylinder, note that you can find it by calculating \( \nabla \phi \), with \( \phi(x, y) = x^2 + y^2 = a^2 \) describing the surface. \( \text{(15 marks)} \)

(ii) A magnetic field is given, in cylindrical polar coordinates \((r, \theta, z)\), as

\[ \mathbf{H} = H_0 r^2 \hat{\theta} / a^2, \]

with \( r \leq a \), where \( H_0 \) and \( a \) are positive constants. The magnetic field vanishes for \( r > a \). Evaluate

\[ \oint_\mathcal{C} \mathbf{H} \cdot dx, \]

where \( \mathcal{C} \) is the circle \( z = 0, r = R \), described in the anticlockwise sense for \( R < a \). \( \text{(10 marks)} \)

End of Question Paper
Mathematical Formulae:

Spherical Polar Coordinates:

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \]
\[ dV = r^2 \sin \theta dr d\theta d\phi \quad \text{(Element of volume)} \]

In the following \( \mathbf{F} = F_1 \hat{r} + F_2 \hat{\theta} + F_3 \hat{\phi} \) (note that \( \hat{r}, \hat{\theta} \) and \( \hat{\phi} \) are unit vectors):

\[ \nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 F_1 \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta F_2 \right) + \frac{1}{r \sin \theta} \frac{\partial F_3}{\partial \phi} \]

and

\[ \nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_1 & r F_2 & r \sin \theta F_3 \end{vmatrix}. \]

Let \( f \) be a scalar function, then the gradient is given by

\[ \nabla f = \nabla^T f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \]

Plane Polar Coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad dA = dx dy = r dr d\theta \]

Vector Calculus:

\[ \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \]
\[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]
\[ \nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} - \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k} \]
\[ \nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
\[ \nabla \times (\nabla \phi) = 0 \]
\[ \nabla \cdot (\nabla \times \mathbf{A}) = 0 \]
\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]
Vectors:

\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \]
\[ \mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k} \]
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \]
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]

Trigonometry:

\[ \sin(\phi \pm \theta) = \sin \phi \cos \theta \pm \cos \phi \sin \theta \]
\[ \cos(\phi \pm \theta) = \cos \phi \cos \theta \mp \sin \phi \sin \theta \]
\[ \tan(\theta \pm \phi) = \frac{\tan \phi \pm \tan \theta}{1 \mp \tan \phi \tan \theta} \]
\[ \sin(2\phi) = 2 \sin \phi \cos \phi \]
\[ \cos(2\phi) = \cos^2 \phi - 1 = 1 - 2 \sin^2 \phi \]
\[ \sin \phi + \sin \theta = 2 \sin \left( \frac{\phi + \theta}{2} \right) \cos \left( \frac{\phi - \theta}{2} \right) \]
\[ \sin \phi - \sin \theta = 2 \cos \left( \frac{\phi + \theta}{2} \right) \sin \left( \frac{\phi - \theta}{2} \right) \]
\[ \cos \phi + \cos \theta = 2 \cos \left( \frac{\phi + \theta}{2} \right) \cos \left( \frac{\phi - \theta}{2} \right) \]
\[ \cos \phi - \cos \theta = 2 \sin \left( \frac{\phi + \theta}{2} \right) \sin \left( \frac{\phi - \theta}{2} \right) \]