SCHOOL OF MATHEMATICS AND STATISTICS

Mathematics III (Electrical)

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall
Registration number from U-Card (9 digits)
to be completed by student
1 (i) (a) State the Cauchy-Riemann equations for an analytic function \( f(z) = u(x, y) + jv(x, y) \).
(b) Verify that the function \( f(z) = z^2 \) satisfies the Cauchy-Riemann equations.

(10 marks)

(ii) Let \( A = \{ z : |z - j| < 2 \} \), \( B = \{ x + jy : y \leq 0 \} \), \( z_0 = 1 - 2j \), \( z_1 = e^{j\pi} \), and \( z_2 = -j \).
(a) Determine which \( z_i \) lie in \( A \), and which \( z_i \) lie in \( B \).
(b) Determine the image of the region \( A \) under the map
\[
f(z) = \frac{z + 1}{z + j}
\]

(15 marks)

2 (i) (a) State Taylor’s theorem for a function \( f(z) \) which is analytic in the disc centered at \( z_0 \) of radius \( R \).
(b) Use Taylor’s theorem to find the first three terms of the Taylor series expansion of the function \( f(z) = \sin z \) about the point \( z_0 = j \).

(12 marks)

(ii) Find the first three terms of the Laurent series expansion of the function
\[
f(z) = \frac{2}{(z + j)(2z - j)}
\] about the point \( z = -j \). Your answer should include the region on which the series expansion is valid.

(13 marks)

3 (i) Let \( L \) be the straight line form 0 to \( 1 + j \). Find
\[
\int_L z^2 dz
\]

(7 marks)

(ii) Let \( C \) be the circle centred at \( 1 + j \) of radius 3. Find
\[
\int_C \frac{e^{-jz}}{z^6} dz
\]

(8 marks)

(iii) Let \( S \) be the square with vertices \( 4 - 4j, 4 + 4j, -4 + 4j, -4 - 4j \). Find
\[
\int_S \frac{z + j}{z^2 + 9} dz
\]

(10 marks)
4  

(i) Use the Laplace transform to solve the differential equation

\[ \frac{d^2 y}{dt^2} - \frac{dy}{dt} - 2y(t) = e^{3t} \]

subject to the initial conditions \( y(0) = 0 \) and \( y'(0) = 1 \).  

(12 marks)

(ii) (a) Find the Fourier transform of \( f(t) = \cos(t) \).

\text{Hint: } \cos(t) = \frac{e^{it} + e^{-it}}{2}

(b) Find the Fourier transform of \( g(t) = e^{it} \cos\left(\frac{\pi t}{2}\right) \).  

(13 marks)

End of Question Paper
Table of Laplace Transforms

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
<th>Region of validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant = ( c )</td>
<td>( \frac{c}{s} )</td>
<td>( Re(s) &gt; 0 )</td>
</tr>
<tr>
<td>( e^{\alpha t} )</td>
<td>( \frac{1}{s-\alpha} )</td>
<td>( Re(s) &gt; \alpha )</td>
</tr>
<tr>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
<td>( Re(s) &gt; 0 )</td>
</tr>
<tr>
<td>( \cos kt )</td>
<td>( \frac{s}{s^2+k^2} )</td>
<td>( Re(s) &gt; 0 )</td>
</tr>
<tr>
<td>( \sin kt )</td>
<td>( \frac{k}{s^2+k^2} )</td>
<td>( Re(s) &gt; 0 )</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
<td>( Re(s) &gt; 0 )</td>
</tr>
<tr>
<td>( t^n e^{\alpha t} )</td>
<td>( \frac{n!}{(s-\alpha)^{n+1}} )</td>
<td>( Re(s) &gt; \alpha )</td>
</tr>
<tr>
<td>( e^{\alpha t} \sin kt )</td>
<td>( \frac{k}{(s-\alpha)^2+k^2} )</td>
<td>( Re(s) &gt; \alpha )</td>
</tr>
<tr>
<td>( e^{\alpha t} \cos kt )</td>
<td>( \frac{s-\alpha}{(s-\alpha)^2+k^2} )</td>
<td>( Re(s) &gt; \alpha )</td>
</tr>
<tr>
<td>( \delta(t - T) )</td>
<td>( e^{-sT} )</td>
<td>delta function</td>
</tr>
<tr>
<td>( H(t - T) )</td>
<td>( \frac{e^{-sT}}{s} )</td>
<td>step function</td>
</tr>
<tr>
<td>( H(t) - H(t - T) )</td>
<td>( \frac{1}{s}(1 - e^{-sT}) )</td>
<td>rectangular pulse</td>
</tr>
</tbody>
</table>

**Note:** in this table the parameters \( \alpha \) and \( k \) are real constants and \( H \) is the Heaviside step function.
Some general properties of the Laplace transform

In the following table the notation \( L\{f(t)\} = F(s) \) has been used.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L{af(t) + bg(t)} = aL{f(t)} + bL{g(t)} )</td>
<td>linearity</td>
</tr>
<tr>
<td>( L{\frac{d}{dt}f(t)} = sF(s) - f(0) )</td>
<td>differentiation w.r.t. ( t )</td>
</tr>
<tr>
<td>( L{\frac{d^2}{dt^2}f(t)} = s^2F(s) - sf(0) - f'(0) )</td>
<td>differentiation twice with respect to ( t )</td>
</tr>
<tr>
<td>If ( g(t) = \int_0^t f(u)du ) then ( L{g(t)} = \frac{1}{s}F(s) )</td>
<td>integration</td>
</tr>
<tr>
<td>( L{tf(t)} = -\frac{dF}{ds} )</td>
<td>differentiation w.r.t. ( s )</td>
</tr>
<tr>
<td>( L{e^{-kt}f(t)} = F(k + s) )</td>
<td>shift</td>
</tr>
<tr>
<td>( L{f(at)} = \frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>( L{f(t - a)H(t - a)} = e^{-as}F(s) )</td>
<td>time delay</td>
</tr>
</tbody>
</table>

**Convolution:** For causal functions

\[
f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau
\]

and has Laplace transform \( F(s)G(s) \).

**Residues**

The formula for the residue at a pole, \( z_0 \), of order 1 is

\[
\lim_{z \to z_0} [(z - z_0)f(z)]
\]

The general formula for the residue at a pole, \( z_0 \), of order \( m \) is

\[
\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]
\]
## Table of Fourier Transforms

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega )</td>
<td>( F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt )</td>
</tr>
</tbody>
</table>

### Standard Functions

\( e^{-at} \) for \( a > 0 \)
\( e^{-at^2} \) for \( a > 0 \)

\[
\Pi(t) = \begin{cases} 
1 & \text{for } |t| < \frac{1}{2} \\
0 & \text{for } |t| > \frac{1}{2} 
\end{cases}
\]

\[
\Delta(t) = \begin{cases} 
1-|t| & \text{for } |t| < 1 \\
0 & \text{for } |t| > 1 
\end{cases}
\]

### Some general results

- **Symmetry**
  \[
  F(\omega) = 2\pi f(-\omega)
  \]

- **Differentiation**
  \[
  j\omega F(\omega)
  \]

- **Time Shift**
  \[
  e^{-j\omega \tau} F(\omega)
  \]

- **Frequency Shift**
  \[
  F(\omega - \theta)
  \]

- **Scaling**
  \[
  \frac{1}{|a|} F\left(\frac{\omega}{a}\right)
  \]

- **Convolution**
  \[
  F(\omega)G(\omega)
  \]

### Generalised functions

- \( 1 \)
- \( \delta(t) \)
- \( \delta(t - \tau) \)
- \( e^{j\beta t} \)
- \( \text{III}\left(\frac{t}{\tau}\right) \)

For the ‘Shah’ function

\[
\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n)
\]