Data Provided: Formula Sheet

The University Of Sheffield.

MAS242

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester
Exam Period 2012-2013

Mathematics III(Electrical) 2 **hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) (a) State the Cauchy-Riemann equations for an analytic function $f(z) = u(x, y) + jv(x, y)$.

(b) Verify that the function $f(z) = z^2$ satisfies the Cauchy-Riemann equations.

(10 marks)

(ii) Let $A = \{z : |z - j| < 2\}$, $B = \{x + jy : y \leq 0\}$, $z_0 = 1 - 2j$, $z_1 = e^{j\frac{\pi}{4}}$, and $z_2 = -j$.

(a) Determine which z_i lie in A, and which z_i lie in B.

(b) Determine the image of the region A under the map

$$f(z) = \frac{z+1}{z+j}$$

(15 marks)

2 (i) (a) State Taylor's theorem for a function $f(z)$ which is analytic in the disc centered at z_0 of radius R.

(b) Use Taylor's theorem to find the first three terms of the Taylor series expansion of the function $f(z) = \sin z$ about the point $z_0 = j$.

(12 marks)

(ii) Find the first three terms of the Laurent series expansion of the function $f(z) = \dfrac{2}{(z+j)(2z-j)}$ about the point $z = -j$. Your answer should include the region on which the series expansion is valid.

(13 marks)

3 (i) Let L be the straight line form 0 to $1 + j$. Find

$$\int_L z^2 dz$$

(7 marks)

(ii) Let C be the circle centred at $1 + j$ of radius 3. Find

$$\int_C \frac{e^{-jz}}{z^6} dz$$

(8 marks)

(iii) Let S be the square with vertices $4 - 4j$, $4 + 4j$, $-4 + 4j$, $-4 - 4j$. Find

$$\int_S \frac{z+j}{z^2+9} dz$$

(10 marks)

4 (i) Use the Laplace transform to solve the differential equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y(t) = e^{3t}$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$. ***(12 marks)***

(ii) (a) Find the Fourier transform of $f(t) = \cos(t)$.

Hint: $\cos(t) = \dfrac{e^{jt} + e^{-jt}}{2}$

(b) Find the Fourier transform of $g(t) = e^{j\pi t}\cos(\frac{\pi t}{2})$. ***(13 marks)***

End of Question Paper

MAS242 FORMULA SHEET

Table of Laplace Transforms

$f(t)$	$F(s)$	Region of validity
constant $= c$	$\frac{c}{s}$	$Re(s) > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$Re(s) > \alpha$
t	$\frac{1}{s^2}$	$Re(s) > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
$e^{\alpha t} \sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$e^{\alpha t} \cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$\delta(t-T)$	e^{-sT}	delta function
$H(t-T)$	$\frac{e^{-sT}}{s}$	step function
$H(t) - H(t-T)$	$\frac{1}{s}(1 - e^{-sT})$	rectangular pulse

Note: in this table the parameters α and k are real constants and H is the Heaviside step function.

Some general properties of the Laplace transform

In the following table the notation $\mathbf{L}\{f(t)\} = F(s)$ has been used.

$\mathbf{L}\{af(t) + bg(t)\} = a\mathbf{L}\{f(t)\} + b\mathbf{L}\{g(t)\}$	linearity
$\mathbf{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathbf{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0) - f'(0)$	differentiation twice with respect to t
If $g(t) = \int_0^t f(u)du$ then $\mathbf{L}\{g(t)\} = \frac{1}{s}F(s)$	integration
$\mathbf{L}\{tf(t)\} = -\frac{dF}{ds}$	differentiation w.r.t. s
$\mathbf{L}\{e^{-kt}f(t)\} = F(k+s)$	shift
$\mathbf{L}\{f(at)\} = \frac{1}{\|a\|}F\left(\frac{s}{a}\right)$	scaling
$\mathbf{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$	time delay

Convolution: For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Residues

The formula for the residue at a pole, z_0, of order 1 is

$$\lim_{z\to z_0}[(z-z_0)f(z)]$$

The general formula for the residue at a pole, z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right).$$

Table of Fourier Transforms

Time Domain		Frequency Domain
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$		$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
Standard Functions		
$e^{-\alpha t}$ for $\alpha > 0$		$\frac{2\alpha}{\alpha^2+\omega^2}$
e^{-at^2} for $a > 0$		$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$\Pi(t) = \begin{cases} 1 & \text{for } \lvert t\rvert<\frac{1}{2} \\ 0 & \text{for } \lvert t\rvert>\frac{1}{2} \end{cases}$		$sinc\frac{\omega}{2} = \begin{cases} \frac{sin(\omega/2)}{\omega/2} & \text{for } \omega\neq 0 \\ 1 & \text{for } \omega=0 \end{cases}$
$\triangle(t) = \begin{cases} 1-\lvert t\rvert & \text{for } \lvert t\rvert<1 \\ 0 & \text{for } \lvert t\rvert>1 \end{cases}$		$sinc^2\frac{\omega}{2}$
Some general results		
$\left.\begin{matrix} f(t) \\ F(t) \end{matrix}\right\}$	symmetry	$\begin{cases} F(\omega) \\ 2\pi f(-\omega) \end{cases}$
$\frac{df}{dt}$	differentiation	$j\omega F(\omega)$
$f(t-\tau)$	time shift	$e^{-j\omega\tau}F(\omega)$
$e^{j\theta t}f(t)$	frequency shift	$F(\omega-\theta)$
$f(at)$	scaling	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
$\int_{-\infty}^{\infty} f(u)g(t-u)du$	convolution	$F(\omega)G(\omega)$
Generalised functions		
1		$2\pi\delta(\omega)$
$\delta(t)$		1
$\delta(t-\tau)$		$e^{-j\omega t}$
$e^{j\theta t}$		$2\pi\delta(\omega-\theta)$
$\text{III}\left(\frac{t}{\tau}\right)$		$\tau\text{III}\left(\frac{\omega\tau}{2\pi}\right)$
for the 'Shah' function	$\text{III}(t) = \sum_{n=-\infty}^{\infty}\delta(t-n)$	